# Electrodynamics

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## 1 Vector Analysis

### 1.1 Vector Algebra

The fundamantal object of Vector analysis is the **vector**. To define what a vector is in a physcal sense, we first introduce the concept of a scalar.

**Definition 1.1: Scalar**

A **scalar** is a quantity that has only magnitude and no direction. Examples are temperature, mass, time, etc.

**Definition 1.2: Vector**

A **vector** is a quantity that has both magnitude and direction. Examples are velocity, force, acceleration, etc. It typically is denoted by a letter with an arrow on top of it, like $\vec{a}$ or $\vec{F}$ when written by hand. In typesetting, it is often written in boldface, like $\boldsymbol{v}$ or $\boldsymbol{F}$.

The length of a vector $\boldsymbol{a}$ is denoted by $|\boldsymbol{a}|$ and is called the **magnitude** of the vector, sometimes simply $a$.

Importantly, vectors do not have a fixed position in space. They can be moved around through space freely.

We now define four operations on vectors that are essential to work with them.

i. **Addition:** The sum of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is defined as placing the tail of $\boldsymbol{b}$ at the head of $\boldsymbol{a}$. Notably, addition is commutative ($\boldsymbol{a}+\boldsymbol{b}=\boldsymbol{b}+\boldsymbol{a}$) and associative ($\boldsymbol{a}+(\boldsymbol{b}+\boldsymbol{c})=(\boldsymbol{a}+\boldsymbol{b})+\boldsymbol{c}$).

ii. **Multiplication by a Scalar:** The product of a vector $\boldsymbol{a}$ and a scalar $k$ is defined as multiplying the magnitude of $\boldsymbol{a}$ by $k$ while keeping the direction of $\boldsymbol{a}$ unchanged. This operation is distributive ($k(\boldsymbol{a}+\boldsymbol{b})=k\boldsymbol{a}+k\boldsymbol{b}$)

iii. **Dot Product:** The dot product is defined as

$$A\cdot B=|\boldsymbol{A}||\boldsymbol{B}|\cos(\theta),$$

where $\theta$ is the angle between the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$. It is commutative ($\boldsymbol{A}\cdot\boldsymbol{B}=\boldsymbol{B}\cdot\boldsymbol{A}$) and distributive ($\boldsymbol{A}\cdot(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\cdot\boldsymbol{B}+\boldsymbol{A}\cdot\boldsymbol{C}$).

iv. **Cross Product:** The cross product is defined as

$$\boldsymbol{A}\times\boldsymbol{B}=AB\sin(\theta)\hat{\boldsymbol{n}}.$$

where $\theta$ is the angle between the two vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ and $\hat{\boldsymbol{n}}$ is the unit vector perpendicular to the plane spanned by $\boldsymbol{A}$ and $\boldsymbol{B}$. The cross product is anti-commutative ($\boldsymbol{A}\times\boldsymbol{B}=-\boldsymbol{B}\times\boldsymbol{A}$) and distributive ($\boldsymbol{A}\times(\boldsymbol{B}+\boldsymbol{C})=\boldsymbol{A}\times\boldsymbol{B}+\boldsymbol{A}\times\boldsymbol{C}$).

Some first applications can be found in the following examples.

**Example 1.3:**

Let $\boldsymbol{C}=\boldsymbol{A}-\boldsymbol{B}$, and calculate the dot product of $\boldsymbol{C}$ with itself.

**Solution.** We start by noticing that by definition, the dot product of a vector with itself is the square of its magnitude. Thus we have

$$\boldsymbol{C}\cdot\boldsymbol{C}=|\boldsymbol{C}|^2.$$

Now we can use the definition of the vector $\boldsymbol{C}$ to rewrite this as

$$\begin{aligned}|\boldsymbol{C}|^2&=|\boldsymbol{A}-\boldsymbol{B}|^2=|\boldsymbol{A}|^2-2\boldsymbol{A}\cdot\boldsymbol{B}+|\boldsymbol{B}|^2\\ C^2&=A^2+B^2-2AB\cos(\theta),\end{aligned}$$

which is simply the law of cosines.

We now want to introduce a more practical way to work with vectors. For this, we will introduce a coordinate system. The most common one is the Cartesian coordinate system. If we let $\hat{x}$, $\hat{y}$ and $\hat{z}$ be the unit vectors in the $x$-, $y$- and $z$-direction respectively, we can write any vector $\boldsymbol{a}$ in the following form

$$\boldsymbol{a}=a_x\hat{x}+a_y\hat{y}+a_z\hat{z}.$$

Using this notation we can rewrite the operations on vectors in a more practical way.

**Theorem 1.4:**

The addition of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ as well as the multiplication by a scalar is done component-wise.

**Proof.** We can write the vectors in component form as

$$\boldsymbol{a}+\boldsymbol{b}=(a_x\hat{x}+a_y\hat{y}+a_z\hat{z})+(b_x\hat{x}+b_y\hat{y}+b_z\hat{z}).$$

Now using associativity and communtativity of addition, we can rearrange this to

$$\boldsymbol{a}+\boldsymbol{b}=(a_x+b_x)\hat{x}+(a_y+b_y)\hat{y}+(a_z+b_z)\hat{z}.$$

The multiplication by a scalar $k$ is done in the same way. □

**Theorem 1.5:**

The dot product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a}\cdot\boldsymbol{b}=a_xb_x+a_yb_y+a_zb_z.$$

**Proof.** We again use the fact we have seen in Example 1.3 that $\boldsymbol{i}\cdot\boldsymbol{i}=|\boldsymbol{i}|^2$. We further notice, that for two perpendicular vectors $\boldsymbol{j}$ and $\boldsymbol{k}$, the dot product is zero, as $\theta=90°$. Accordingly,

$$\begin{aligned}\boldsymbol{a}\cdot\boldsymbol{b}&=(a_x\hat{x}+a_y\hat{y}+a_z\hat{z})\cdot(b_x\hat{x}+b_y\hat{y}+b_z\hat{z})\\&=a_xb_x+a_yb_y+a_zb_z.\end{aligned}$$

□

**Theorem 1.6:**

The cross product of two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ is given by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}.$$

**Proof.** We make use of the fact, that for any vector $\boldsymbol{v}$, $\boldsymbol{v} \times \boldsymbol{v} = \boldsymbol{0}$. Furthermore, for two vector $\boldsymbol{u} \perp \boldsymbol{v}$, we have $|\boldsymbol{u} \times \boldsymbol{v}| = uv$. Doing the computation of the determinant, we get

$$\begin{aligned} \boldsymbol{a} \times \boldsymbol{b} &= (a_x\hat{x} + a_y\hat{y} + a_z\hat{z}) \times (b_x\hat{x} + b_y\hat{y} + b_z\hat{z}) \\ &= (a_yb_z - a_zb_y)\hat{x} + (a_zb_x - a_xb_z)\hat{y} + (a_xb_y - a_yb_x)\hat{z} \end{aligned}$$

which can be written in the determinant form. □

As the cross product yields a vector itself, we can also take the dot product or cross product the result with another vector. This leads to the following

**Theorem 1.7: Scalar triple product**

The scalar triple product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by

$$\boldsymbol{a} \cdot (\boldsymbol{b} \times \boldsymbol{c}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}.$$

The resulting number represents the volume of the parallelepiped spanned by the three vectors. Notice that we can get the same result by evaluating $b \cdot (c \times a)$ or $c \cdot (a \times b)$. Furthermore, we can also exchange the dot and the cross product, when beeing cautious about the placement of the brackets.

**Theorem 1.8: Vector triple product**

The vector triple product of three vectors $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ is given by

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}).$$

This is also known as the **BAC-CAB** rule.

**Proof.** This can be most easily shown by writing out both sides of the equation in component form. □

Notice that $(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c}$ yields a completely different result.

All higher products of vectors can be derived from the by repeated application of the above theorems.

**Example 1.9:**

Prove that

$$[\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})] + [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a})] + [\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})] = \boldsymbol{0}.$$

**Solution.** We apply the BAC-CAB rule to each term in the sum.

$$\begin{aligned} [\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c})] &= \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b}) \\ [\boldsymbol{b} \times (\boldsymbol{c} \times \boldsymbol{a})] &= \boldsymbol{c}(\boldsymbol{b} \cdot \boldsymbol{a}) - \boldsymbol{a}(\boldsymbol{b} \cdot \boldsymbol{c}) \\ [\boldsymbol{c} \times (\boldsymbol{a} \times \boldsymbol{b})] &= \boldsymbol{a}(\boldsymbol{c} \cdot \boldsymbol{b}) - \boldsymbol{b}(\boldsymbol{c} \cdot \boldsymbol{a}) \end{aligned}$$

Making use of the commutativity of the dot product, we can see that every term cancels out, which leaves us with the null vector $\boldsymbol{0}$.

For the further sections it will be useful to have a vector describe a point in space. For this, we introduce the **position vector** which is typically denoted by $\boldsymbol{r}$.

**Definition 1.10: Position Vector**

The position vector $\boldsymbol{r}$ of a point in space is defined as the vector from the origin to that point. It is given by

$$\boldsymbol{r} = x\hat{\boldsymbol{x}} + y\hat{\boldsymbol{y}} + z\hat{\boldsymbol{z}}$$

in a Cartesian coordinate system.

Another useful concept will be the infinitesimal vector $\mathrm{d}\boldsymbol{l}$ from $(x, y, z)$ to $(x + \mathrm{d}x, y + \mathrm{d}y, z + \mathrm{d}z)$.

In electrodynamics, one will often encounter a system with a source point, $\boldsymbol{r}'$, and a field point, $\boldsymbol{r}$. The vector from the source point to the field point is called the **separation vector** and is denoted by $\boldsymbol{\imath} := \boldsymbol{r} - \boldsymbol{r}'$.

## 1.2 Differential Calculus

Consider a function $f(x)$. What does the derivative $\frac{\mathrm{d}f}{\mathrm{d}x}$ mean? It tells us how rapidly the function $f$ changes when we change the argument $x$ by a small amount $\mathrm{d}x$.

Suppose now, that we have a function like Temperature $T(x, y, z)$ which depends on three variables. We want to generalize the concept of the derivative to this case. To do this, we use a theorem on partial derivatives which states

$$\mathrm{d}T = \frac{\partial T}{\partial x}\mathrm{d}x + \frac{\partial T}{\partial y}\mathrm{d}y + \frac{\partial T}{\partial z}\mathrm{d}z.$$

We can rewrite this in form of a dot product as

$$\mathrm{d}T = \nabla T \cdot \mathrm{d}\boldsymbol{l},$$

where $\nabla T$ is the **gradient** of the function $T$.

The importance of the gradient that it points in the direction of the steepest ascent of the function $T$, with magnitude equal to the rate of change of the function in that direction.

The gradient has a formal appearance of a vector, $\nabla$, "muliplied" with the function $T$. This $\nabla$ is called the **nabla operator** or **del operator**. It is defined as

$$\nabla = \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.$$

It is important to state, that the nabla operator is not a vector in the usual sense, but rather an operator that acts on functions to produce vectors. This vector like appearence is what motivates to analyze the behavior of the nabla operator when applied to dot and cross products of vector functions.

**Definition 1.11: Divergence**

The **divergence** of a vector field $\boldsymbol{F}$ is defined as

$$\nabla \cdot \boldsymbol{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}.$$

It measures the "outflow" of the vector field at a point.

**Definition 1.12: Curl**

The **curl** of a vector field $\boldsymbol{F}$ is defined as

$$\nabla \times \boldsymbol{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}.$$

It measures the "rotation" of the vector field at a point.

As with usual derivatives, there exist product rules for the nabla operator. They are listed below.

1. $$\nabla(fg) = f\nabla g + g\nabla f$$

2. $$\begin{aligned}\nabla(\boldsymbol{F}\cdot\boldsymbol{G}) =& \boldsymbol{F}\times(\nabla\times\boldsymbol{G}) + \boldsymbol{G}\times(\nabla\times\boldsymbol{F}) \\ &+ (\boldsymbol{F}\cdot\nabla)\boldsymbol{G} + (\boldsymbol{G}\cdot\nabla)\boldsymbol{F}\end{aligned}$$

3. $$\nabla\cdot(f\boldsymbol{a}) = f(\nabla\cdot\boldsymbol{a}) + \boldsymbol{a}\cdot(\nabla f)$$

4. $$\nabla\cdot(\boldsymbol{a}\times\boldsymbol{b}) = \boldsymbol{b}\cdot(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\cdot(\nabla\times\boldsymbol{b})$$

5. $$\nabla\times(f\boldsymbol{a}) = f(\nabla\times\boldsymbol{a}) - \boldsymbol{a}\times(\nabla f)$$

6. $$\begin{aligned}\nabla\times(\boldsymbol{a}\times\boldsymbol{b}) =& \boldsymbol{a}(\nabla\cdot\boldsymbol{b}) - \boldsymbol{b}(\nabla\cdot\boldsymbol{a}) \\ &+ (\boldsymbol{b}\cdot\nabla)\boldsymbol{a} - (\boldsymbol{a}\cdot\nabla)\boldsymbol{b}\end{aligned}$$

In similar fashion, we can also define the second derivatives of a function. The most important thing to notice is the following

**Theorem 1.13:**

The curl of the gradient such as the divergence of the curl of a function is always zero.

$$\nabla\times(\nabla f) = \boldsymbol{0} \quad \text{and} \quad \nabla\cdot(\nabla\times\boldsymbol{F}) = 0.$$

**Proof.** For the second statement, we can use the definition of the curl and the divergence

$$\begin{aligned}\nabla\cdot(\nabla\times\boldsymbol{F}) &= \nabla\cdot\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \\ &= \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}\end{aligned}$$

As the first two rows are equal, the determinant is zero.

For the first statement, we can also use the definitions and see that

$$\nabla\times(\nabla f) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}.$$

As the second and third row are linearly dependent, the determinant is zero again. □

The more interesting results are the **Laplacian** and the **gradient of the divergence**.

**Definition 1.14: Laplacian**

The **Laplacian** of a function $f$ is defined as

$$\nabla^2 f = \nabla\cdot(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Sometimes, one also speaks of the Laplacian of a vector field $\boldsymbol{F}$, which is defined as

**Definition 1.15: Laplacian of a Vector Field**

The **Laplacian** of a vector field $\boldsymbol{F}$ is defined as

$$\nabla^2\boldsymbol{F} = \nabla(\nabla\cdot\boldsymbol{F}) - \nabla\times(\nabla\times\boldsymbol{F}),$$

or in the case of a Cartesian coordinate system

$$\nabla^2\boldsymbol{F} = (\nabla^2 F_x)\hat{x} + (\nabla^2 F_y)\hat{y} + (\nabla^2 F_z)\hat{z}.$$

As we've seen this definition already includes the divergence of a gradient.

## 1.3 Integral Calculus

In electrodynamics, we will often encounter the follwoing integrals.

**Definition 1.16: Line Integral**

The **line integral** of a vector field $\boldsymbol{F}$ along a curve $\mathcal{C}$ is defined as

$$\int_a^b \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}.$$

It is common to use the notation $\oint$ if $a = b$ and the curve is closed. A classic example of this is the work integral of a force $\boldsymbol{F}$ along a path $\mathcal{C}$, which is given by $W = \int_{\mathcal{C}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{l}$.

Normally, the value of this integral is critically dependent on the path taken. However, for special functions $\boldsymbol{C}$, the value of the integral is independent of the path. These functions are called **conservative vector fields**.

**Definition 1.17: Surface Integral**

The **surface integral** of a vector field $\boldsymbol{F}$ over a surface $\mathcal{S}$ is defined as

$$\iint_{\mathcal{S}} \boldsymbol{F}\cdot\mathrm{d}\boldsymbol{A}.$$

Here, $\mathrm{d}\boldsymbol{A}$ is the area element of the surface $\mathcal{S}$ with direction perpendicular to the surface.

This quantity is often also called the **flux** of the vector field $\boldsymbol{F}$ through the surface $\mathcal{S}$ as if the vector field $\boldsymbol{F}$ represents a flow of fluid for example, the surface integral describes how much fluid flows through the surface $\mathcal{S}$.

**Definition 1.18: Volume Integral**

The **volume integral** of a scalar function $f$ over a volume $\mathcal{V}$ is defined as

$$\iiint_{\mathcal{V}} f\,\mathrm{d}\tau.$$

Sometimes one might also encounter the volume integral of a vector function $\boldsymbol{F}$, which is defined as

$$\iiint_{\mathcal{V}} \boldsymbol{F} \mathrm{d}\tau = \hat{x} \iiint_{\mathcal{V}} F_x \mathrm{d}\tau + \hat{y} \iiint_{\mathcal{V}} F_y \mathrm{d}\tau + \hat{z} \iiint_{\mathcal{V}} F_z \mathrm{d}\tau.$$

To actaully calculate these integrals, we need the fundamental theorem of calculus.

**Theorem 1.19: Fundamental Theorem of Calculus**

Let $f$ be a function that is continuous on the interval $[a, b]$ and differentiable on the open interval $(a, b)$. Then

$$\int_a^b f'(x) \mathrm{d}x = f(b) - f(a).$$

As we have some more derivatives at our disposal, we can also define a fundamental theorem for all of them.

**Theorem 1.20: Fundamental theorem for gradients**

Let $f$ be a function that is continuous on the volume $\mathcal{V}$ and differentiable on the interior of $\mathcal{V}$. Then

$$\int_b^a \nabla f \cdot \mathrm{d}\boldsymbol{l} = f(\boldsymbol{b}) - f(\boldsymbol{a}).$$

An important conclusion from this is, that the gradient of a function is a conservative vector field as the right hand side of the equation only depends on the endpoints $\boldsymbol{a}$ and $\boldsymbol{b}$, but not on the path taken.

**Theorem 1.21: Gauss' Theorem**

Let $\boldsymbol{F}$ be a vector field that is continuous on the volume $\mathcal{V}$ and differentiable on the interior of $\mathcal{V}$. Then

$$\iint_{\partial\mathcal{V}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{A} = \iiint_{\mathcal{V}} \nabla \cdot \boldsymbol{F} \mathrm{d}\tau.$$

Visually, this theorem states that if we have some sources of a fluid in the volume $\mathcal{V}$, the total outflow of the fluid through the surface $\partial\mathcal{V}$ can be calculated either by evaluating the surface integral or by summing up the sources of the fluid in the volume $\mathcal{V}$.

**Theorem 1.22: Stokes' Theorem**

Let $\boldsymbol{F}$ be a vector field that is continuous on the surface $\mathcal{S}$ and differentiable on the interior of $\mathcal{S}$. Then

$$\oint_{\partial\mathcal{S}} \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{l} = \iint_{\mathcal{S}} \nabla \times \boldsymbol{F} \cdot \mathrm{d}\boldsymbol{A}.$$

This essentially means, that the circulation of a vector field $\boldsymbol{F}$ around the boundary $\partial\mathcal{S}$ of a surface $\mathcal{S}$ is equal to the flux of the curl of the vector field through the surface $\mathcal{S}$.

This theorem further implies that the flux of the curl of a vector field through a surface is only dependent on the boundary of the surface, but not on the surface itself.

## 1.4 Curvilinear Coordinates

For some problems, it is more convenient to use a spherical or cylindrical coordinate system. In the case of a spherical coordinate system, we define the coordinates $(r, \theta, \phi)$ as in figure 1.

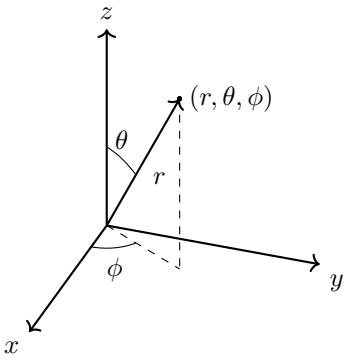


Figure 1: Spherical Coordinates

When we want to integrate over a volume in spherical coordinates, we have to take into account the volume element in spherical coordinates, which is given by the Jacobian determinant of the transformation from Cartesian to spherical coordinates.

**Definition 1.23: Jacobian Determinant**

The **Jacobian determinant** of a transformation from Cartesian coordinates $(x, y, z)$ to spherical coordinates $(r, \theta, \phi)$ is given by

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}.$$

The volume element in spherical coordinates is then given by $\mathrm{d}\tau = J \mathrm{d}r\, \mathrm{d}\theta\, \mathrm{d}\phi$.

Doing the calculation for our spherical coordinates, we get that

$$J = r^2 \sin(\theta).$$

In similar fashion, we can define cylindrical coordinates $(r, \phi, z)$, where $r$ is the distance from the $z$-axis, $\phi$ is the angle in the $xy$-plane and $z$ is the height above the $xy$-plane depicted in figure 2. Here, $J = r$

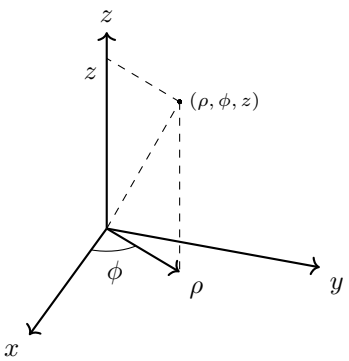


Figure 2: Cylindrical Coordinates

# 2 Electrostatics

The fundamental problem in Electrodynamics, is to find the Force on a test charge $Q$ based on some source charges $q_1, q_2, \dots$ where all these objects can be in motion. To deal with this, the perhaps most essential concept is the following:

**Theorem 2.1: Superposition Principle**

The force on a test charge $Q$ is the sum of the forces exerted by each source charge $q_i$.

Using the superposition principle, we know that the interaction between two charges is completely independent of any other charges present in the system. We therefore can treat the forces between each pair of charges separately and then sum them up to get the total force on the test charge $Q$.

Before we start with moving charges, we just consider stationary source charges. This is the special case of Electrodynamics is called **Electrostatics**.

## 2.1 Coulomb's Law

The key result in Electrostatics from which everything else follows is the **Coulomb's Law**, which is a purely experimental result.

**Theorem 2.2: Coulomb's Law**

If we have a source charge $q$ and a test charge $Q$, the force on $Q$ is

$$F = \frac{1}{4\pi\varepsilon_0}\frac{qQ}{\imath^2}\hat{\boldsymbol{\imath}}.$$

The constant $\varepsilon_0$ is called the *permittivity of free space*. It's value is given by

$$\varepsilon_0 = 8.85 \times 10^{-12}\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}.$$

$\boldsymbol{\imath}$ denotes the separation vector from $r'$(Location of the source charge) to $r$(Location of the test charge).

$$\boldsymbol{\imath} = \mathbf{r} - \mathbf{r}'.$$

It is easy to see, that the Coulomb force is repulsive if the charges have the same sign and attractive if they have opposite signs.

## 2.2 Electric Field

If we have many source charges, the total Force on $Q$ looks like the following:

$$\begin{aligned}\boldsymbol{F} = \boldsymbol{F}_1 + \boldsymbol{F}_2 + \dots &= \frac{1}{4\pi\varepsilon_0}\left(\frac{Qq_1}{\imath_1^2}\hat{\boldsymbol{\imath}}_1 + \frac{Qq_2}{\imath_2^2}\hat{\boldsymbol{\imath}}_2 + \dots\right)\\ &= \frac{1}{4\pi\varepsilon_0}Q\left(\frac{q_1}{\imath_1^2}\hat{\boldsymbol{\imath}}_1 + \frac{q_2}{\imath_2^2}\hat{\boldsymbol{\imath}}_2 + \dots\right).\end{aligned}$$

We notice, that the term in brackets is essentially just one big sum. We can define a new quantity called the **Electric Field** $\boldsymbol{E}$.

**Definition 2.3: Electric Field**

The Electric Field $\boldsymbol{E}$ at a point $\boldsymbol{r}$ is defined as

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{\imath_i^2}\hat{\boldsymbol{\imath}}_i.$$

For simplicity the factor $\frac{1}{4\pi\varepsilon_0}$ is already included in the electric field. For our net force on $Q$ we can now write

$$\boldsymbol{F} = Q\boldsymbol{E}.$$

Pretty neat right? The Electric field only depends on the position of the source charges and not on the test charge. This turns out very useful as if we know the electric field, we can calculate the force on any test charge $Q$ at any position. An example of the electric field is given in figure 3. It shows the electric field of a point charge.

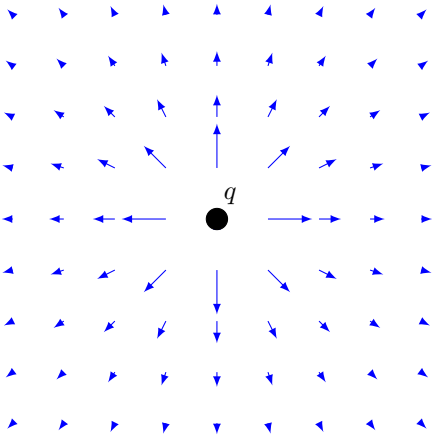


Figure 3: Electric Field of a Point Charge

### 2.2.1 Continuous Charge Distributions

The power of the electric field really shines when we have continuous charge distributions. Our definition of the electric field still holds, but we have to replace the sum with an integral.

**Definition 2.4: Electric Field for Continuous Charge Distributions**

The Electric Field $\boldsymbol{E}$ at a point $\boldsymbol{r}$ due to a continuous charge distribution is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\int \frac{\mathrm{d}q}{\imath^2}\hat{\boldsymbol{\imath}}.$$

In order to get the infinitesimal charge $\mathrm{d}q$, we need to know how the charge is distributed in space. The three classic cases are, **line charges** with line charge density $\lambda(\boldsymbol{r})$, surface charges with surface charge density $\sigma(\boldsymbol{r})$ and volume charges with volume charge density $\rho(\boldsymbol{r})$. The infinitesimal charge is then given by

$$\mathrm{d}q = \lambda(\boldsymbol{r'})\mathrm{d}l' = \sigma(\boldsymbol{r'})\mathrm{d}a' = \rho(\boldsymbol{r'})\mathrm{d}\tau'.$$

It is also worth mentioning that $\boldsymbol{\imath}$ still denotes the separation vector from $dq$ to $r$.

## 2.3 Divergence and Curl of Electrostatic Fields

In principle, we are done with Electrostatics. However, the integral for the electric field can be quite difficult. Especially as it is a vector integral so we always need to take care of the direction of the field aswell. In the following pages we will develop some tools to make our life easier.

### 2.3.1 Divergence of the Electrostatic Field

The first tool is the Divergence of the Electric Field. To find properties about it, let us once again start with a single point charge $q$ at the origin. The electric field is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}.$$

One can sketch this vector field. If we now smooth out the image, we can just connect the vectors to form what is called **field lines**. The density of field lines is proportional to the magnitude of the electric field. Notice, that for this to work, we need to think of the field in three dimensions. A simple example can be seen in figure 4.

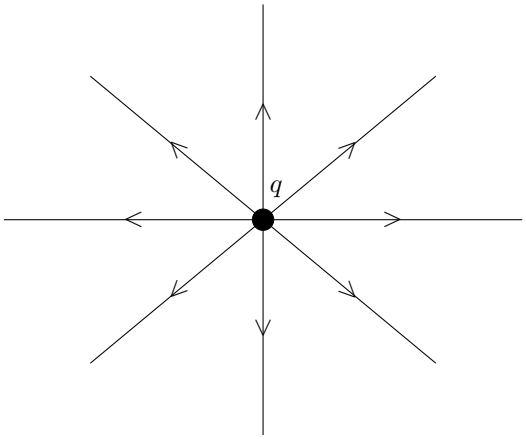


Figure 4: Field Lines of a Point Charge

In a system of charges the qualitative behavior of the field lines should be the same. For example, $2q$ should have twice as many field lines as $q$. The field lines begin at positive charges and end at negative charges. The only other possibility for field lines is go to infinity. Also, the lines can never cross as if that were the case, the field would have two directions at the same point. To quantify this, we introduce the concept of the **Electric Flux**.

**Definition 2.5: Electric Flux**

The Electric Flux $\Phi_E$ through a surface $\mathcal{S}$ is defined as

$$\Phi_E = \iint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a}.$$

It measures the "amount" of field lines passing through the surface.

We are now ready to derive the first of the four Maxwell equations, which is called **Gauss' Law**.

**Theorem 2.6: Gauss' Law**

The total electric flux through a closed surface $\mathcal{S}$ is equal to the total charge enclosed by the surface divided by $\varepsilon_0$.

$$\oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

**Proof.** Let us first consider a single point charge $q$ at the origin. The electric flux through a sphere of radius $r$ centered at the origin is given by

$$\Phi_E = \oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \iint \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{\boldsymbol{r}}\cdot r^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi\hat{\boldsymbol{r}}$$

Notably, the $r^2$ term cancels out. Evaluating the integrals, we obtain

$$\oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \frac{q}{\varepsilon_0}.$$

Assume now, that we didn't have a sphere, but any closed surface $\mathcal{S}$. The amount of field lines passing through the surface is still the same as the field lines are not dependent on the shape of the surface. The only thing that matters is the total charge enclosed by the surface. Therefore, the above equation holds for any closed surface $\mathcal{S}$.

If we have multiple charges $q_1, q_2, \ldots$, we can apply the superposition principle.

$$E = \sum_i E_i.$$

For a surface enclosing all the charges, the total flux is then given by

$$\oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \sum_i \oiint_{\mathcal{S}} \boldsymbol{E}_i\cdot\mathrm{d}\boldsymbol{a} = \sum_i \frac{q_i}{\varepsilon_0} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

□

This is the integral form of Gauss' Law. It is often the suitable form to use. Nonetheless, it is quite simple to turn it into a differential form.

**Theorem 2.7: Differential Form of Gauss' Law**

The electric flux through a closed surface $\mathcal{S}$ is equal to the integral of the divergence of the electric field over the volume $\mathcal{V}$ enclosed by the surface.

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}.$$

**Proof.** We start with the integral form of Gauss' Law

$$\oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \frac{Q_{\text{enc}}}{\varepsilon_0}.$$

Applying the Divergence Theorem(1.21), we can rewrite the left hand side as

$$\oiint_{\mathcal{S}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{a} = \iiint_{\mathcal{V}} \nabla\cdot\boldsymbol{E}\mathrm{d}\tau.$$

The total charge enclosed by the surface can be rewritten as

$$Q_{\text{enc}} = \iiint_{\mathcal{V}} \rho\mathrm{d}\tau.$$

Combining these two equations and using the fact that this holds for any volume $\mathcal{V}$, we can equate the integrands to get

$$\nabla\cdot\boldsymbol{E} = \frac{\rho}{\varepsilon_0}.$$

□

This is a very general law and does not restrict on the assumptions of Electrostatics. It is also valid in the case of Electrodynamics.[1]

When problems have some sort of symmetry involved, Gauss' Law will solve the problem in just a few lines. For

[1] The above proof may seem a bit flawed with the usage of field lines. A more rigorous proof would follow from applying the Divergence operator to the formula for the electric field directly, and applying the Dirac Delta function.

application you want to find a **Gaussian surface**, which is a surface on which the electric field is perpendicular and constant. There are three main symmetries we consider:

- **Spherical Symmetry:** A spherically symmetric charge distribution implies a spherically symmetric electric field. ⇒ Use a concentric sphere as Gaussian surface.
- **Cylindrical Symmetry:** A cylindrically symmetric charge distribution implies a cylindrically symmetric electric field. ⇒ Use a cylinder as Gaussian surface.
- **Planar Symmetry:** A planar symmetric charge distribution implies a planar symmetric electric field. ⇒ Use a box as Gaussian surface.

In the latter examples, notice that the electric field is perpendicular to the surface of the Gaussian surface on the sides. The above examples can also be combined with superposition.

### 2.3.2 Curl of the Electrostatic Field

**Theorem 2.8: Curl of the Electrostatic Field**

The curl of the electrostatic field is zero.

$$\nabla \times \boldsymbol{E} = 0.$$

**Proof.** Let us again look at a single point charge at the origin. The electric field is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{\boldsymbol{r}^2}\hat{\boldsymbol{r}}.$$

Let us calculate the line integral of the electric field along a path from $A$ to $B$. In spherical coordinates, we get

$$E \cdot \mathrm{d}l = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\mathrm{d}r.$$

This works, as $E$ is radial and therefore only depends on the distance $r$ from the origin. Integrating the above expression, we get

$$\int_a^b \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = \frac{1}{4\pi\varepsilon_0}q\int_a^b \frac{1}{r^2}\mathrm{d}r = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_a}-\frac{1}{r_b}\right).$$

For a closed path, we have $a = b$ and therefore the line integral is zero.

$$\oint \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = 0.$$

Applying Stokes' Theorem (1.22), we can rewrite this as

$$\nabla \times \boldsymbol{E} = 0.$$

Since the Curl operator is distributive, this also holds for a system of multiple charges. □

## 2.4 Electric Potential

What's quite cool about the electrostatic field is that it's curl is zero. We will use the fact, that any vector field with zero curl can be written as the gradient of a scalar field. This is what defines the **Electric Potential**.

**Definition 2.9: Electric Potential**

The Electric Potential $V$ at a point $\boldsymbol{r}$ is defined as

$$V(\boldsymbol{r}) = -\int_\infty^{\boldsymbol{r}} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}.$$

Now, in principle one could have chosen any reference point for the potential. However, it is common practice to set the potential at infinity to zero. This is called the **Zero Potential at Infinity**. The key exception to this is, when our charges itself go to infinity (for example an infinite line charge). In this case, we have to choose a different reference point.

A cute analogy to the electric potential is the gravitational potential. In fact we can make use of a visual analogy to gravity, if we let positive charges be like hills and negative charges be like valleys. An example of this is given in figure 5.

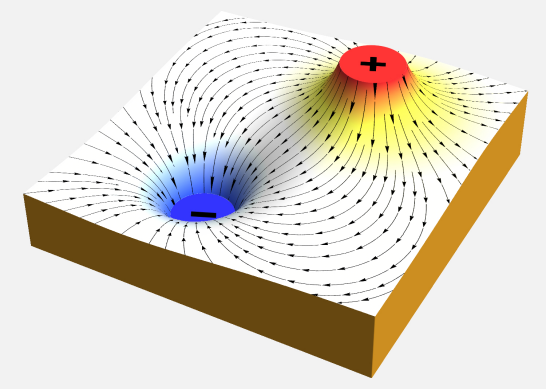


Figure 5: Analogy of Electric Potential to Gravity

The differential form of the potential is given by the following theorem.

**Theorem 2.10: Electric Field from Electric Potential**

The electric field is the negative gradient of the electric potential.

$$\boldsymbol{E} = -\nabla V.$$

**Proof.** If we compute the potential difference for two points $A$ and $B$, we get

$$V_B - V_A = -\int_\infty^{\boldsymbol{r}_B} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} + \int_\infty^{\boldsymbol{r}_A} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l} = -\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}.$$

Meanwhile, the fundamental theorem of gradients (Theorem 1.20) tells us that

$$V_B - V_A = -\int_{\boldsymbol{r}_A}^{\boldsymbol{r}_B} \nabla V\cdot\mathrm{d}\boldsymbol{l}.$$

Comparing these two equations and noticing that they hold for any two points $A$ and $B$, we can equate the integrands to get

$$\boldsymbol{E} = -\nabla V.$$

□

Notice, that the electric potential also obeys the superposition principle. It's similar to the Electric field, but now we are summing scalars instead of vectors which is a lot nicer.

If we take the divergence of the potential, we get the same result as for the electric field.

**Theorem 2.11: Poisson's Equation**

The Laplacian of the electric potential is equal to the negative charge density divided by $\varepsilon_0$.

$$\nabla\cdot(\nabla V) = \nabla^2 V = -\frac{\rho}{\varepsilon_0}.$$

We shall return to Poisson's Equation later in section 3.

Now from a problem solving perspective, it is not really useful to find the potential only after finding the electric field. So let us look at how we can find it directly. Poisson's Equation gives us the charge density if we know the potential. So we will reverse engineer this to get the following result

**Theorem 2.12: Electric potential for a continuous charge distribution**

The electric potential at a point $\boldsymbol{r}$ due to a continuous charge distribution is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\rho(\boldsymbol{r'})}{\boldsymbol{\imath}} \mathrm{d}\tau'.$$

To do this we will first find the electric potential of a point charge

**Lemma 2.13: Electric Potential of a Point Charge**

The electric potential at a point $\boldsymbol{r}$ due to a point charge $q$ at the origin is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{\boldsymbol{\imath}}.$$

**Proof.** The potential of a point charge is given by

$$V(\boldsymbol{r}) = -\int_{\infty}^{\boldsymbol{r}} \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l}.$$

Using spherical coordinates, only the radial component of the electric field matters, so we can write

$$V(\boldsymbol{r}) = -\int_{\infty}^{\boldsymbol{r}} E_r \mathrm{d}r = -\int_{\infty}^{\boldsymbol{r}} \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2} \mathrm{d}r = \frac{1}{4\pi\varepsilon_0} \frac{q}{r}.$$

□

**Proof.** [Proof of Theorem 2.12] We start by applying the superposition principle to our lemma.

$$V(\boldsymbol{r}) = \sum_i V_i(\boldsymbol{r}) = \sum_i \frac{1}{4\pi\varepsilon_0} \frac{q_i}{\boldsymbol{\imath}_i}.$$

Transforming this into an integral, we get

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \int \frac{\mathrm{d}q}{\boldsymbol{\imath}}.$$

Which can be rewritten as wanted by using the definition of the infinitesimal charge $\mathrm{d}q$. □

The key advantage with this compared to the electric field is that the unit vector $\hat{\boldsymbol{\imath}}$ is gone as the potential is a scalar field. As a warning, we assume that the potential is zero at infinity. For all the artificial problems that extend to infinity, our integrals diverge.

As a last comment for now, whenever we cross a surface charge $\sigma$, the electric field undergoes a discontinuity. To calculate the jump in the electric field, we can use Gauss' Law just above and below the surface. We find

$$\boldsymbol{E}^{\perp}_{\text{above}} - \boldsymbol{E}^{\perp}_{\text{below}} = \frac{\sigma}{\varepsilon_0} \hat{n}.$$

The potential meanwhile is continuous across the surface charge.

## 2.5 Work and Energy in Electrostatics

If we want to move a test charge $Q$ from point $A$ to point $B$ in an electric field, we have to apply a force to overcome the electric force. It is given by $F = -Q\boldsymbol{E}$. The work done is then

$$W = -\int_A^B Q\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = Q \int_A^B \nabla V \cdot \mathrm{d}\boldsymbol{l} = Q(V_A - V_B).$$

As this does not depend on the path taken, we call the electrostatic force **conservative**. If we bring a charge $Q$ from infinity to a point $r$ in an electric field, the work done is

$$W = -\int_{\infty}^{r} Q\boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = QV(r).$$

This quantity is called the **Potential Energy** of the charge $Q$ at point $r$.

If we analyze a system of charges, we get the follwoing formula.

**Theorem 2.14: Energy of a System of Point Charges**

The total work needed to assemble a system of $n$ point charges $q_1, q_2, \ldots, q_n$ is given by

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V_i.$$

Where $V_i$ is the potential at the position of charge $q_i$ due to all other charges.

**Proof.** Let us bring in the charges one by one from infinity to their respective positions. The work done here is given by

$$W = \frac{1}{4\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j>i}^{n} \frac{q_i q_j}{\boldsymbol{\imath}_{ij}}.$$

If we instead count every pair twice and then divide by two, we get

$$W = \frac{1}{8\pi\varepsilon_0} \sum_{i=1}^{n} \sum_{j\neq i}^{n} \frac{q_i q_j}{\boldsymbol{\imath}_{ij}} = \frac{1}{2} \sum_{i=1}^{n} q_i \left( \sum_{j\neq i}^{n} \frac{1}{4\pi\varepsilon_0} \frac{q_j}{\boldsymbol{\imath}_{ij}} \right).$$

The term in brackets is just the potential at the position of charge $i$ due to all other charges. We can therefore rewrite the above equation as

$$W = \frac{1}{2} \sum_{i=1}^{n} q_i V_i.$$

□

If we instead want to analyze the work needed to assemble a system with a continuous charge distribution, we need to alter our formula slightly. Simply replacing the sum with an integral yields

$$W = \frac{1}{2} \iiint_{\mathcal{V}} \rho V \mathrm{d}\tau. \tag{1}$$

Sometimes it might be more useful to use the following theorem

**Theorem 2.15: Energy of a continuous charge distribution**

The total energy of a continuous charge distribution is given by

$$W = \frac{\varepsilon_0}{2} \iiint_{\mathcal{V}} E^2 \mathrm{d}\tau.$$

**Proof.** We start by applying Gauss' Law to equation (1).

$$W = \frac{\varepsilon_0}{2} \iiint_{\mathcal{V}} (\nabla \cdot \boldsymbol{E})\, V \mathrm{d}\tau.$$

Applying integration by parts yields

$$W = \frac{\varepsilon_0}{2}\left(-\iiint_{\mathcal{V}} \boldsymbol{E}\cdot(\nabla V)\mathrm{d}\tau + \oiint_{\mathcal{S}} V\boldsymbol{E}\cdot\mathrm{d}a\right).$$

Using the fact that $\nabla V = -\boldsymbol{E}$, we can rewrite this as

$$W = \frac{\varepsilon_0}{2}\left(\iiint_{\mathcal{V}} E^2\mathrm{d}\tau + \oiint_{\mathcal{S}} V\boldsymbol{E}\cdot\mathrm{d}a\right).$$

The above equation is correct, as long as we enclose all the charges in the volume $\mathcal{V}$. If we choose our volume $\mathcal{V}$ to be all of space, the surface integral goes to zero.[2] Our equation then simplifies to

$$W = \frac{\varepsilon_0}{2}\iiint_{\text{all space}} E^2\mathrm{d}\tau.$$

□

It is worth noticing that theorem 2.14 and theorm 2.15 are standing for two completely diffrent things. In fact, simply by looking at a point charge at the origin, we can see that the two equations are not equivalent.

This is because theorem 2.15 includes the energy needed to create the charges in the first place and as a result, will diverge if the total charge in a system isn't zero. The difference between theese two equations comes from the fact that under the integral we also include our infinitesimal charge $\mathrm{d}q$ in the potential $V$.

## 2.6 Conductors

An **Isolator** is a material where electrons are restricted from moving away from a given atom. A **Conductor** on the other hand allows electrons to move around freely. An ideal conductor would have an unlimited supply of free electrons. This of course cannot be achieved in practice, but metals do come quite close.

Consider placing a conductor in an external electric field. The electrons in the conductor will move towards one side of the conductor and create a electric field themselves. Exactly until inside the conductor the total electric field is zero. The implications are summarized as follows

**Theorem 2.16: Properties of Conductors**

For conductors the following properties hold:

- $\boldsymbol{E} = 0$ The electric field inside a conductor is zero.
- $\rho = 0$ The charge density inside a conductor is zero.
- $V = \text{const}$. The potential inside a conductor is constant.

One can also notice, that any net charge on a conductor must reside on its surface as there simply isn't any space inside it. For calculations it is also useful to know that the electric field just outside a conductor is perpendicular to thesurface.

A neat effect of conductors happens when you place a charge near a conductor. The charges in the conductor will then move around to cancel out the electric field inside the conductor. This is called **Induced Charge**.

If we place some charge $q$ inside a cavity of a conductive sphere, a charge $-q$ will be induced on to the inner surface of the sphere. As the total charge inside the conductor is zero, a net charge $q$ resides on the surface of our sphere. Summarizing the effects, the electric field inside the sphere is zero. The electric field outside the sphere is the same as if all the charge was concentrated in the center of the sphere by Gauss' Law.

**Theorem 2.17: Faraday's Cage**

The electric field inside a cavity of a conductor is zero if there are no charges inside the cavity.

**Proof.** Let us assume the contrary, that there is a non-zero electric field inside the cavity. If we then integrate along a closed path such as in figure 6, we see that the integral must be strictly positive (or negative depending on the direction of the electric field) inside the cavity, whilst beeing zero inside the conductor.

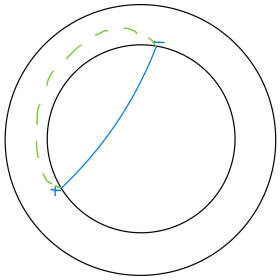

Figure 6: Electric field inside a cavity

This means that $\oint \boldsymbol{E}\cdot d\boldsymbol{l}$ is not zero, which is a contradiction to the electrostatic theory. □

If one can determine the electric field or the potential at the surface of a conductor, one can determine the charge distribution on the surface by using the equation

$$-\frac{\partial V}{\partial n} = \boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}.$$

In the presence of a charge, the conductor will experience a force per unit area given by $f = \sigma\boldsymbol{E}$. The electric field however is discontinuous at the surface of the conductor. In order to use the above equation, one needs to use the average of just above and just below the surface.

**Theorem 2.18: Force on a charged conductor**

The force per unit area on a charged conductor in an electric field is given by

$$f = \frac{\sigma^2}{2\varepsilon_0}\hat{\boldsymbol{n}}.$$

This also defines a pressure on the conductor.[a]

$$P = \frac{\varepsilon_0}{2}\boldsymbol{E}^2.$$

[a] This follows from the above by applying $\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}}$

## 2.7 Capacitors

If we have two conductors with charges $+Q$ and $-Q$, the potential difference between the two conductors is given by

$$V = V_+ - V_- = -\int_{(-)}^{(+)} \boldsymbol{E}\cdot\mathrm{d}\boldsymbol{l}.$$

[2] The electric field decreases with distance as $1/r^2$, potential with $1/r$ while the area only increases with $r^2$.

We do not want to calculate $\boldsymbol{E}$ as this would be a nightmare to do for any non-trivial geometry. Instead we notice that $\boldsymbol{E}$ is proportional to Q. Because of this, $V$ is also proportional to $Q$. We can define this proportionality constant as follows

**Definition 2.19: Capacitance**

The capacitance $C$ of a system of conductors is defined as

$$C = \frac{Q}{V}.$$

It is determined purely by the geometry of our two conductors.

For the sake of completeness, usually we take $Q$ as the positive charge and $V$ as the positive potential difference.

If we want to charge up a capacitor we need to remove electrons from one plate and add them to the other. The work needed for this is given by

$$W = \int \mathrm{d}W = \int_0^Q \frac{q}{C}\mathrm{d}q = \frac{Q^2}{2C} = \frac{1}{2}CV^2.$$

A special type of capacitor is the **Parallel Plate Capacitor**. It consists of two parallel plates with a distance $d$ between them. An example of this is given in figure 7. The electric field between the plates is homogeneous and given by

$$\boldsymbol{E} = \frac{\sigma}{\varepsilon_0}\hat{\boldsymbol{n}} = \frac{Q}{\varepsilon_0 A}\hat{\boldsymbol{n}}.$$

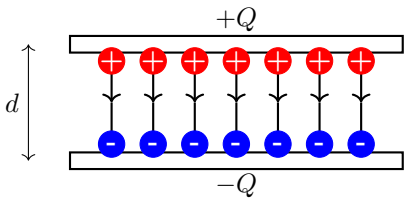


Figure 7: A plate capacitor with a homogeneous electric field between the plates

# 3 Potential Deep Dive

We will now return to the previously derived Poisson's Equation (2.11).

$$\nabla^2 V = -\frac{\rho}{\varepsilon_0}.$$

If we look at volume without any charges, we get the following equation

**Theorem 3.1: Laplace's Equation**

In regions where there is no charge, the potential satisfies Laplace's Equation

$$\nabla^2 V = 0.$$

This equation is probably the most fundamental equation when it comes to solving electrostatic problems. Written out in Cartesian coordinates, it looks like

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0.$$

The solutions to Laplace's Equation are called *Harmonic Functions*. There is no general form of them except when we look at the one dimensional case. In this case, the general solution is given by

$$V(x) = mx + b.$$

for some, by boundary conditions, determined constants $m$ and $b$.

In all cases, there is the following property of harmonic functions:

**Theorem 3.2: Mean Value Property**

The value of a harmonic function at the center of a sphere is equal to the average of the values on the surface of the sphere.

**Proof.** We will do the proof in three dimensions. To do this, consider a sphere with radius $R$ centered at the origin and a charge $q$ placed outside the sphere on the $z$ axis.

The potential on a point $\boldsymbol{r}$ on the surface of the sphere is given by

$$V = \frac{1}{4\pi\varepsilon_0}\frac{q}{\boldsymbol{\imath}},$$

where $\boldsymbol{\imath}$ can be expressed by the law of cosines as

$$\boldsymbol{\imath}^2 = z^2 + R^2 - 2zR\cos(\theta).$$

The average value of the potential can be found by the following integral

$$\begin{aligned}
\langle V\rangle &= \frac{1}{4\pi R^2}\iint_S V\,\mathrm{d}a \\
&= \frac{1}{4\pi R^2}\frac{q}{4\pi\varepsilon_0}\iint_S \frac{1}{\sqrt{z^2+R^2-2zR\cos(\theta)}}R^2\sin(\theta)\mathrm{d}\theta\mathrm{d}\phi \\
&= \frac{q}{4\pi\varepsilon_0}\frac{1}{2zR}\sqrt{z^2+R^2-2zR\cos(\theta)}\Big|_{\theta=0}^{\theta=\pi} \\
&= \frac{1}{4\pi\varepsilon_0}\frac{q}{z}.
\end{aligned}$$

This however is exactly the potential at the center of the sphere. By applying the superposition principle, we can extend this to any charge distribution outside the sphere. □

This property also implies that there cannot be a local Maxima or Minima inside a volume and must reside on the boundary.

Unfortunately Laplace's Equation is not enough to solve a problem by itself. As we have seen before in the one dimensional case, we need suitable boundary conditions. There is quite a rich palette of options to choose from when deciding on boundary conditions. Normally they are served as a **Uniqueness Theorem**. The two most common and useful of these are the following

**Theorem 3.3: First Uniqueness Theorem**

The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely determined if the potential $V$ is specified on the boundary $\mathcal{S}$.

**Proof.** Let us assume there are two solutions $V_1$ and $V_2$ to Laplace's equation for some volume. If we introduce the the difference

$$V_3 = V_1 - V_2,$$

we notice that $V_3$ must also satisfy Laplace's Equation.

But clearly $V_3$ is zero on the boundary $\mathcal{S}$, as both $V_1$ and $V_2$ are equal on the boundary. By the Mean Value Property (3.2), we can conclude that $V_3$ must be zero everywhere within the volume $\mathcal{V}$, which contradicts the assumption that $V_1$ and $V_2$ are different. □

One can actually strengthen this result and use Poisson's Equation instead of Laplace's Equation. We then require additionally that the charge distribution $\rho$ is specified in the volume $\mathcal{V}$.

Having specified boundary conditions can be achieved quite simply, by having a conductor as the boundary, which is often connected to ground ($V = 0$). But sometimes, we know charges $Q_A$, $Q_B$ and so on some conductors. We maybe also know the charge density $\rho$ between the conductors. In this case we can make use of the following theorem

**Theorem 3.4: Second Uniqueness Theorem**

The solution to Laplace's Equation in some volume $\mathcal{V}$ surrounded by conductors and containing a specified charge density $\rho$ is uniquely determined if the total charge on each conductor is specified.

The proof for this theorem is mathematically quite invovled and we will thus skip it.

## 3.1 The method of Images

The **Method of Images** is a technique used to find an electrostatic potential using the first uniqueness theorem (3.3) without having to solve Laplace's equation directly. It is best explained with an example.

**Example 3.5:**

Suppose we hold a point charge $q$ at a distance $d$ from an infinite grounded conducting plane. What is the potential at a point $P$ above the plane?

**Solution.** A useful strategy to such problems is to first write down the boundary conditions. In our case, we have the following

- V(z=0) = 0: The potential on the plane is zero.
- $V \to 0$ as $z \to \infty$: The potential at infinity is zero.

Solving Laplace's Equation directly with these boundary conditions will be quite difficult. So let's instead look at a seemingly different problem. Place a charge $q$ at a distance $d$ above the $xy$-plane and a charge $-q$ at a distance $d$ below it. The potential at a point $P$ is then given by theorem 2.13 as

$$V(\boldsymbol{r}) = \frac{\left(\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right)}{4\pi\varepsilon_0}.$$

But what does this have to do with our original problem? Lets look at value of this potential at the $z = 0$ and at $z \to \infty$. We notice that in both cases, the potential is zero. So we have the same boundary conditions. As the charge distribution is also the same above the plane, by the first uniqueness theorem, the solution must be the same. The resulting field is displayed in figure 8.

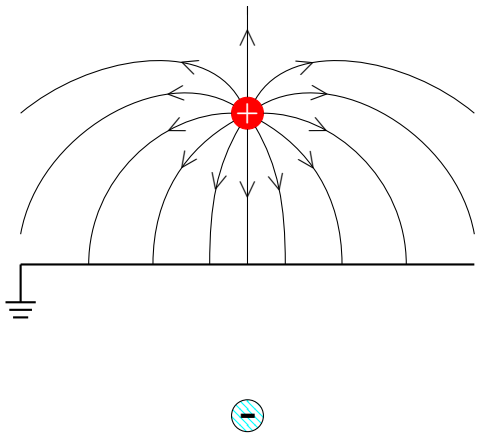

Figure 8: The image charge setup for a point charge above a grounded plane

From here, its an easy step to solve for the induced charge density on the plane. As seen in Section 2.6, the electric field just above the plane is given by

$$\sigma = -\varepsilon_0 \frac{\partial V}{\partial n}.$$

In the case of our problem, we would get

$$\sigma(x, y) = \frac{-qd}{2\pi(x^2 + y^2 + d^2)^{3/2}}.$$

If we want to find the total induced charge, we can simply integrate over the entire plane. After doing the math we find

$$Q_{\text{induced}} = \iint_{\mathcal{S}} \sigma \mathrm{d}a = -q.$$

Which is what one would expect.

If we want to find the force excerted on the charge $q$ by the plane, we can again make use of the image charge setup as the potential around $q$ is the same in both cases. The force for our problem is then given by

$$F = -\frac{1}{4\pi\varepsilon_0}\frac{q^2}{(2d)^2}\hat{\boldsymbol{z}}.$$

As a last note: Energy is NOT the same in both situations. Since the potential on the ground is zero, it is also not contributing to the work done when moving around the charge $q$ in contrast to the image charge.

The method of images is a very powerful tool as it can also be used on more than just one charge point. The tricky thing about it is finding the right setup of image charges.

## 3.2 The direct way to solve Laplace's Equation

We can solve Laplace's Equation directly by using separation of variables assuming that either the potential $V$ or the charge density $\sigma$ is specified on the boundaries of some volume in which we need to find the potential. The basic strategy relies on finding solutions that are products of functions, each of which depends on only one of the coordinates. Since the math is somewhat non-trivial we will look at an example in cartesian coordinates to illustrate the method.

**Example 3.6:**

Two infinite grounded conducting planes, both parallel to the $xz$-plane are placed at $y = 0$ and $y = a$. The left end at $x = 0$ is closed of with an infinite strip with potential $V_0(y)$. What is the potential in the enclosed region?

**Solution.** We start by noticing that the problem is symmetric in the $z$-direction. Laplace's Equation thus reduces to

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0.$$

By the first uniqueness theorem we have a unique solution with the boundary conditions

i. $V(x, 0) = 0$ (grounded plane)
ii. $V(x, a) = 0$ (grounded plane)
iii. $V(0, y) = V_0(y)$ (potential on the left end)
iv. $V(x, y) \to 0$ as $x \to \infty$ (potential at infinity)

The first step to solving such a problem is to look for solutions in the form of

$$V(x, y) = X(x)Y(y).$$

This will most likely not yield the correct solution, but by cleverly combining solutions of this type later on, we can solve our original problem. Plugging this into Laplace's Equation, we get

$$Y\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + X\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = 0.$$

By separating the variables we can rewrite this as

$$\frac{1}{X}\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + \frac{1}{Y}\frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = f(X) + g(Y) = 0.$$

Notice that $f$ and $g$ have to be constant functions, as otherwise we could keep one variable constant and vary the other one, which would alter $f(X) + g(Y)$. We thus have two seperate equations

$$\frac{\mathrm{d}^2 X}{\mathrm{d}x^2} = CX \quad \text{and} \quad \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -CY.$$

Let us assume that $C$ is positive[3] and define $k^2 = C$. The solution to the first equation is then

$$X(x) = A\mathrm{e}^{kx} + B\mathrm{e}^{-kx}.$$

The second equation yields

$$Y(y) = C\sin(ky) + D\cos(ky).$$

Together we get a solution of the form

$$V(x, y) = \left(A\mathrm{e}^{kx} + B\mathrm{e}^{-kx}\right)\left(C\sin(ky) + D\cos(ky)\right).$$

Applying the fourth boundary condition, we see that $A = 0$ and applying the first boundary condition, we get $D = 0$. Rewriting $B \cdot C$ as $C$ we get

$$V(x, y) = \mathrm{e}^{-kx}C\sin(ky).$$

With the second boundary condition we see that

$$k = \frac{n\pi}{a} \text{ for } n \in \mathbb{N}$$

[3] We will later see why this is sensical.

Unless $V_0(y)$ just happens to be of the form $\sin(n\pi y/a)$, we cannot satisfy the third boundary condition. However, we can make use of the linearity of Laplace's Equation.[4]

$$V(x, y) = \sum_{n=1}^{\infty} C_n \mathrm{e}^{-n\pi x/a}\sin(n\pi y/a).$$

The question now is, cab we find coefficients $C_n$ for any given function $V_0(y)$? The answer is yes, as this is a Fourier series.

## 3.3 Multipole Expansion

Multipole expansion is a technique used to approximate the potential of a charge configuration at a large distance. It becomes useful when the total charge of a system is zero and the approximation that the potential is just zero is not enough. The idea is best demonstrated by the electrical dipole.

**Example 3.7: Physical electric dipole**

Find the potential far away of a physical electric dipole consisting of two point charges $q$ and $-q$ located at $x = \frac{d}{2}$ and $x = -\frac{d}{2}$ respectively.

**Solution.**

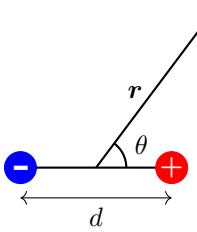


Figure 9: A physical electric dipole consisting of two point charges

Letting $\theta$ be defined as in figure 9, we can write the distance between the point charges and the point $\boldsymbol{r}$ as

$$\boldsymbol{r}_{\pm} = r^2 + \left(\frac{d}{2}\right)^2 \mp rd\cos(\theta) = r^2\left(1 \mp \frac{d}{r}\cos(\theta) + \frac{d^2}{4r^2}\right).$$

For large distances, we have $r \gg d$ and thus simplify

$$\boldsymbol{r}_{\pm}^2 \approx r^2\left(1 \mp \frac{d}{r}\cos(\theta)\right).$$

Since ultimately we care about the value $\frac{1}{\boldsymbol{r}}$, we can invert the term and then apply the binomial expansion[5]

$$\frac{1}{\boldsymbol{r}_{\pm}} \approx \frac{1}{r}\left(1 \mp \frac{d}{r}\cos(\theta)\right)^{-\frac{1}{2}} \approx \frac{1}{r}\left(1 \pm \frac{d}{2r}\cos(\theta)\right).$$

Plugging this into the potential of a point charge, we get

$$V(\boldsymbol{r}) \approx \frac{1}{4\pi\varepsilon_0}\frac{qd\cos\theta}{r^2}.$$

Using similar ideas one can generalize this idea. For example a system of two dipoles is a quadrupole and so on. The potential of such a construct then scales with $1/r^n$ where $n$ is the order of the multipole.[6]

Sometimes we want to look at a more general case than

[4] This means that if $V_1$ and $V_2$ are solutions, then $\alpha V_1 + \beta V_2$ is also a solution.

[5]
$$(1 + x)^n \approx 1 + nx.$$

[6] This means for a Monopole it scales with $1/r$, for a Dipole with $1/r^2$, a Quadrupole with $1/r^3$ and so on.

just the physical dipole. Here, the following theorem is useful

**Theorem 3.8: Multipole Expansion**

The potential of a charge distribution is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{n=0}^{\infty}\frac{1}{r^{n+1}}\iiint_{\mathcal{V}}(r')^n\,P_n(\cos\theta)\rho(\boldsymbol{r'})\mathrm{d}\tau'.$$

Where $P_n$ are the Legendre polynomials and $\theta$ is the angle between the position vector $\boldsymbol{r}$ and the position vector $\boldsymbol{r'}$ of the charge element $\mathrm{d}\tau'$.

**Proof.** The potential at a point $\boldsymbol{r}$ is given by theorem 2.12

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\iiint_{\mathcal{V}}\frac{\rho(\boldsymbol{r'})}{\scriptscriptstyle\nwarrow}\mathrm{d}\tau'.$$

Using the law of cosines (1.3), we can write $\scriptscriptstyle\nwarrow$ as

$$\scriptscriptstyle\nwarrow = r\sqrt{1+\varepsilon} \text{ with } \varepsilon = \left(\frac{r'}{r}\right)\left(\frac{r'}{r} - 2\cos\theta\right).$$

We can write the taylor expansion of $\frac{1}{\scriptscriptstyle\nwarrow}$ as

$$\frac{1}{\scriptscriptstyle\nwarrow} = \frac{1}{r}\left(1 - \frac{1}{2}\varepsilon + \frac{3}{8}\varepsilon^2 - \dots\right).$$

Combining likewise terms, we can rewrite this as

$$\frac{1}{\scriptscriptstyle\nwarrow} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_n(\cos\theta).$$

Which concludes the proof. □

### 3.3.1 Monopoles and Dipoles

The equation derived in theorem 3.8 is not that useful as it is. It is mainly thought to be useful as an approximation for the potential of a charge distribution at large distances. Normally we care about the first non-zero terms in the expansion.

The simplest case is where we have a net charge $Q$ that is not zero. In this case, the expression is dominated by the monopole term, which is given by

$$V_{\text{mon}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}.$$

If the total charge is zero, the monopole term vanishes and we have to consider the dipole term. It is given by

**Theorem 3.9: Potential of a Dipole**

The potential of a dipole at a point $\boldsymbol{r}$ is given by

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^2}.$$

Where $\boldsymbol{p}$ is the dipole moment of our charge configuration as defined by definition 3.10.

**Proof.** From theorem 3.8 we can see that the dipole term is given by

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\iiint_{\mathcal{V}} r'\cos\theta\rho(\boldsymbol{r'})\mathrm{d}\tau'.$$

As $\theta$ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r'}$,

$$r'\cos\theta = \hat{\boldsymbol{r}}\cdot\boldsymbol{r'}.$$

This simplifies our integral to

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{1}{r^2}\hat{\boldsymbol{r}}\cdot\iiint_{\mathcal{V}}\boldsymbol{r'}\rho(\boldsymbol{r'})\mathrm{d}\tau'.$$

It is quite common to now define the **Dipole Moment** which completes the proof. □

**Definition 3.10: Dipole Moment**

The dipole moment $\boldsymbol{p}$ of a charge distribution is defined as

$$\boldsymbol{p} = \iiint_{\mathcal{V}}\boldsymbol{r'}\rho(\boldsymbol{r'})\mathrm{d}\tau'.$$

For a so called **Physical Dipole** we have two charges a distant $d$ apart, with equal and opposite charges, $\pm q$. The dipole moment is then given by

$$\boldsymbol{p} = q\boldsymbol{d}.$$

Where $\boldsymbol{d}$ is the vector pointing from the negative to the positive charge. Since $\boldsymbol{p}$ is a vector, we can add different dipole moments together. If we have four alternating charges arranged in a square for example no net dipole moment exists and the quadrupole moment is the lowest order moment that is non-zero. However we will not look at this here.

### 3.3.2 The electric field of a dipole

To close the discussion on electric dipoles, we will look at the electric field of a perfect dipole ($d\to 0, q\to\infty$) placed at the origin and pointing in the $z$-direction. The electric field is then given by

**Theorem 3.11: Electric Field of a Dipole**

The electric field $\boldsymbol{E}$ of a dipole with dipole moment $\boldsymbol{p}$ at a point $\boldsymbol{r}$ is given by

$$\boldsymbol{E}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\left(\frac{3(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}\right).$$

**Proof.** Without loss of generality, let us assume that the dipole is oriented along the $z$-axis. Theorem 3.9 now reads

$$V_{\text{dip}}(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta}{r^2}.$$

The electric field is given according to theorem 2.10 is given by

$$\boldsymbol{E}(\boldsymbol{r}) = -\nabla V_{\text{dip}}(\boldsymbol{r}).$$

This gradient is most conveniently calculated in spherical coordinates

$$E_r = -\frac{\partial V_{\text{dip}}}{\partial r}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{2p\cos\theta}{r^3}\hat{r} = \frac{1}{4\pi\varepsilon_0}\frac{2\boldsymbol{p}\cdot\hat{\boldsymbol{r}}}{r^3}\hat{r}.$$

$$E_\theta = -\frac{1}{r}\frac{\partial V_{\text{dip}}}{\partial\theta}\hat{\theta} = \frac{1}{4\pi\varepsilon_0}\frac{p\sin\theta}{r^3}\hat{\theta}.$$

Now by construction, we know that $\boldsymbol{p}\perp\hat{\boldsymbol{\phi}}$ and as such

$$\boldsymbol{p} = p\cos\theta\hat{\boldsymbol{r}} - p\sin\theta\hat{\boldsymbol{\theta}},$$

where the minus sign is due to the fact the fact that $p$ points in the positive $z$-direction. We can thus rewrite

$$E_\theta = \frac{1}{4\pi\varepsilon_0}\frac{p\cos\theta\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3} = \frac{1}{4\pi\varepsilon_0}\frac{(\boldsymbol{p}\cdot\hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{p}}{r^3}.$$

Combining the two components, we get the electric field of a dipole. As this form doesn't depend on any specific coordinate system, we can justify our assumption at the beginning. □

Again, for a multipole of higher order, the electric field scales with $1/r^{n+1}$ where $n$ is the order of the multipole.[7]

[7] Again, orders scale as powers of two.

# 4 Electric Fields in Matter

So far we looked at electrostatics in vacuum and conductors. But what happens, when we look at an insulator (also called dielectric)? All electrons are tied to their respective atoms and cannot move freely. However, if the electrons align properly, there are some tiny effects that scale up to macroscopic effects. We will now look at the electric field in matter.

## 4.1 Polarization

Imagine placing a neutral atom inside an electric field. The electrons and nucleus will be slightly ripped apart (Figure 10). This creates a very tiny dipole moment. The atom is then said to be **polarized**. If the electric field is not too large, the dipole moment is approximately proportional to the electric field.

$$\boldsymbol{p} = \alpha \boldsymbol{E}.$$

The proportionality constant $\alpha$ is called the **atomic polarizability**. Notie how we make use of the fact that atoms are extremely small and thus we can neglect the monopole term.

For molecules, the situation is a bit more complicted. In general, the polarizability can differ in every direction. In its most general form, the polarizability is given by a second rank tensor called the **polaizability tensor** $\boldsymbol{\alpha}$.

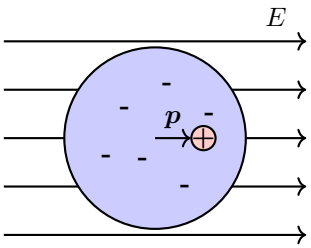


Figure 10: A polarized atom in an electric field

### 4.1.1 Polar Molecules

Some molecules already have a built in permanent dipole moment. This is for example the case with water molecules. Such molecules are called **polar molecules**. The question now is, what happens when we place such a molecule in in an electric field. In the case of a homogenous electric field it is given by

**Theorem 4.1:**

A polar molecule with dipole moment $\boldsymbol{p}$ in a homogenous electric field $\boldsymbol{E}$ experiences a torque given by

$$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}.$$

**Proof.** Since the field is uniform, the forces on the positive and negative ends cancel out. However if we look at the torque, we see that it is

$$\boldsymbol{\tau} = (\boldsymbol{r}_+ \times \boldsymbol{F}_+) + (\boldsymbol{r}_- \times \boldsymbol{F}_-).$$

Letting $\boldsymbol{d} = \boldsymbol{r}_+ - \boldsymbol{r}_-$ be the vector pointing from the negative to the positive end of the dipole, we can rewrite this as

$$\boldsymbol{\tau} = [(\boldsymbol{d}/2) \times (q\boldsymbol{E})] + [(-\boldsymbol{d}/2) \times (-q\boldsymbol{E})] = q\boldsymbol{d} \times \boldsymbol{E} = \boldsymbol{p} \times \boldsymbol{E}.$$

□

Notice that the torque gives rise to an orientation of the dipole moment to align with the electric field.

If we do not have a uniform electric field, the situation is a bit more complicated. Then there is also a force acting on the dipole, which is given by

$$\boldsymbol{F} = (\boldsymbol{p} \cdot \nabla)\boldsymbol{E}.$$

### 4.1.2 Polarization of more than one atom

We can now answer what happens, when we place a dielectric in an electric field. All atoms will either induce a field pointing in the direction of the applied field, or molecules will align themselves with the field. The material is said to be **polarized**. The most convenient metric is the follwoing

**Definition 4.2: Polarization**

The **Polarization** $\boldsymbol{P}$ of a dielectric is defined as the dipole moment per unit volume. It is given by

$$\boldsymbol{P} = \frac{\sum_i \boldsymbol{p}_i}{\Delta V}.$$

For now, let us ignore how the polarization got there and just focus on the field produced by it.

## 4.2 The field of a polarized material

As usual, it is often simpler to look at the potential. For a polarized object, it is given by

**Theorem 4.3: Potential of a Polarized Material**

The potential $V$ at a point $\boldsymbol{r}$ due to a polarized material with polarization $\boldsymbol{P}$ is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \oiint_{\mathcal{S}} \frac{\sigma_b(\boldsymbol{r'})}{\mathscr{r}} \mathrm{d}a + \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\rho_b(\boldsymbol{r'})}{\mathscr{r}} \mathrm{d}\tau'.$$

Where $\sigma_b = \boldsymbol{P} \cdot \hat{\boldsymbol{n}}$ is the bound surface charge density and $\rho_b = -\nabla \cdot \boldsymbol{P}$ is the bound volume charge density.

**Proof.** By theorem 3.9 we know the potential of a single dipole at a point $\boldsymbol{r}$ is given by

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \frac{\boldsymbol{p} \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2}.$$

Naturally, we can extend this to a continuous distribution of dipoles by simply integrating

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{\boldsymbol{P}(\boldsymbol{r'}) \cdot \hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2} \mathrm{d}\tau'.$$

Noticing that

$$\nabla' \left(\frac{1}{\mathscr{r}}\right) = \frac{\hat{\boldsymbol{\mathscr{r}}}}{\mathscr{r}^2},$$

we can turn this into

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \boldsymbol{P} \cdot \nabla' \left(\frac{1}{\mathscr{r}}\right) \mathrm{d}\tau'.$$

Integration by parts, applying the product rule and the divergence theorem gives us

$$V(\boldsymbol{r}) = \frac{1}{4\pi\varepsilon_0} \oiint_{\mathcal{S}} \frac{\boldsymbol{P}}{\mathscr{r}} \cdot \mathrm{d}a - \frac{1}{4\pi\varepsilon_0} \iiint_{\mathcal{V}} \frac{1}{\mathscr{r}} \left(\nabla' \cdot \boldsymbol{P}\right) \mathrm{d}\tau'.$$

We notice, that the first term looks sort of like the potential of a surface charge and the second one like the potential of a volume charge. We can now apply the definitions from the theorem and conclude. □

[8] The prime on the $\nabla$ is there to indicate that we differentiate with respect to the source coordinates.

Using the notation of bound charges is extremely handy, as we can apply all our tricks from chapter 3 to find the potential of a polarized material. The only difference is that we have to first find the bound charge densities $\sigma_b$ and $\rho_b$.

Lastly, it is important to note that the bound charge densities are genuine existing charges. They are not simply a mathematical trick to find the potential.

## 4.3 Electric Displacement

As we have seen, the effect of polarization produces accumulations of bound charge $\rho_b$ inside the dielectric and $\sigma_b$ on the surface. We are now ready to put the field of the bound charges and the field due to everything else (also called **Free charge** $\rho_f$) together.

For this, we define the **electric displacement** $\boldsymbol{D}$.

**Definition 4.4: Electric Displacement**

The electric displacement $\boldsymbol{D}$ is defined as

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P},$$

where $\boldsymbol{E}$ is the electric field due to all charges, including the polarization.[a]

[a] The addition may seem a bit counterintuitive, but it follows from the fact that the electric field goes from positive to negative charges, while the polarization goes from negative to positive charges.

Within the dielectric, the total charge density is

$$\rho = \rho_f + \rho_b.$$

We can thus transform Gauss' Law into yet another form.

**Theorem 4.5: Gauss' Law in Matter**

Gauss' Law in matter is given by

$$\nabla \cdot \boldsymbol{D} = \rho_f.$$

Or in integral form

$$\oiint_S \boldsymbol{D} \cdot \mathrm{d}a = Q_{\text{free enclosed}}.$$

**Proof.** Gauss' Law reads

$$\varepsilon_0 \nabla \cdot \boldsymbol{E} = \rho_b + \rho_f = -\nabla \cdot \boldsymbol{P} + \rho_f.$$

Combining the two divergence terms and applying definition 4.4 we get

$$\nabla \cdot \boldsymbol{D} = \rho_f.$$

Which translates to the integral form as usual. □

This form of Gauss' Law is very useful, as it allows us to only consider the free charges, which is the stuff we control. A typical example is that we know $\rho_f$ and initially do not know $\rho_b$. For appropriate symmetries, we can then immediately use Gauss' Law.

## 4.4 Linear Dielectrics

Let us turn back to the field that creates the polarization. For many substances, the polarization is proportional to the electric field, provided $\boldsymbol{E}$ is not too strong. We can then define the proportionality constant

**Definition 4.6: Electric Susceptibility**

The **Electric Susceptibility** $\chi_e$ of a material is defined as the proportionality constant between the polarization $\boldsymbol{P}$ and the electric field $\boldsymbol{E}$.

$$\boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}.$$

Unfortunately the electric field used for defining the susceptibility is the total electric field, including the field due to the polarization. However, we get the following relation

$$\boldsymbol{D} = \varepsilon_0 (1 + \chi_e) \boldsymbol{E} \coloneqq \varepsilon \boldsymbol{E}.$$

This new constant $\varepsilon$ is called the **Permittivity** of the material. Sometimes also the **relative permittivity** $\varepsilon_r$ is used

$$\varepsilon_r = 1 + \chi_e = \frac{\varepsilon}{\varepsilon_0}.$$

Inside a dielectric, the bound charge density is given by

$$\rho_b = -\nabla \cdot \boldsymbol{P} = -\nabla \cdot \left( \varepsilon_0 \frac{\chi_e}{\varepsilon_r} \boldsymbol{E} \right) = -\left( \frac{\chi_e}{1 + \chi_e} \right) \rho_f.$$

So unless there is free charge in our dielectric, all our bound charges are found on the surface and we can apply Laplace's Equation to find the potential as described in section 3.2. For this it may be useful to have the boundary conditions in a way that only references the free charge.

**Theorem 4.7:**

For the boundary of a linear dielectric, the following holds

$$\varepsilon_{\text{above}} \boldsymbol{E}_{\text{above}} - \varepsilon_{\text{below}} \boldsymbol{E}_{\text{below}} = \sigma_f.$$

Or in terms of the potential

$$\varepsilon_{\text{above}} \frac{\partial V_{\text{above}}}{\partial n} - \varepsilon_{\text{below}} \frac{\partial V_{\text{below}}}{\partial n} = -\sigma_f.$$

## 4.5 Energy in dielectric systems

The energy to assemble a system including a dielectric is given by the following theorem.

**Theorem 4.8: Energy of a System of Charges in a Dielectric**

The total energy of a system of charges in a linear dielectric is given by

$$W = \frac{1}{2} \iiint_{\mathcal{V}} \mathbf{D} \cdot \mathbf{E} \mathrm{d}\tau.$$

**Proof.** Suppose we have a dielectric fixed in place and slowly bring a bit of free charge $\Delta\rho_f$. The polarization and thus the bound charges will change, however we are only intereseted in the work done on $\Delta\rho_f$.

$$\Delta W = \iiint_{\mathcal{V}} (\Delta\rho_f) V \mathrm{d}\tau.$$

Applying Gauss' Law in matter (theorem 4.5) we can rewrite this as

$$\Delta W = \iiint_{\mathcal{V}} [\nabla \cdot (\Delta \boldsymbol{D})] V \mathrm{d}\tau = \iiint_{\mathcal{V}} \nabla[(\Delta \boldsymbol{D}) V] + (\Delta \boldsymbol{D}) \cdot \boldsymbol{E} \mathrm{d}\tau.$$

Applying the divergence theorem (1.21) we can rewrite the first term as a surface integral which goes to zero as we integrate over all of space.

$$\Delta W = \iiint_{\mathcal{V}} (\Delta \boldsymbol{D}) \cdot \boldsymbol{E} \mathrm{d}\tau.$$

So far, this holds for all media. However in a linear dielectric, we have $D = \varepsilon \boldsymbol{E}$ (definition 4.4). Thus assuming infinitesimal $\Delta W$

$$(\Delta \boldsymbol{D}) \cdot \boldsymbol{E} = \varepsilon (\Delta \boldsymbol{E}) \cdot \boldsymbol{E} = \frac{1}{2} \Delta (\varepsilon E^2) = \frac{1}{2} \Delta (\boldsymbol{D} \cdot \boldsymbol{E}).$$

Summing up $\Delta W$ over all infinitesimal changes yields the theorem. □

It is important to note, that the equation in theorem 2.15 is not the same as the one in theorem 4.8. The first one answers the question of how much energy is needed to bring all charges, including the bound charges, into the system. The second one answers the question of how much energy is needed to bring in the free charges. Notice that this includes the energy needed to polarize the dielectric.

As one noramlly only cares about the free charges, the second equation is typically the one you want to use.

## 4.6 Forces on Dielectrics

As we have seen in section 2.6, a conductor gets an induced charge in the presenece of another charge which typically casues a force on the conductor. Similarly, a dielectric can be affected by an external electric field.

Consider for example a parallel plate capacitor with a dielectric partially inserted. The fringing field at the edge of the capacitor will cause a force on the dielectric pulling it into the capacitor.

**Theorem 4.9: Force on a dielectric in a capacitor**

The force on a dielectric in a capacitor is given by

$$F = \frac{1}{2} V^2 \frac{\mathrm{d}C}{\mathrm{d}x}.$$

Where $\mathrm{d}C/\mathrm{d}x$ is the change in capacitance when the dielectric is moved by a distance $\mathrm{d}x$.

**Proof.** The work needed to move the dielectric by a distance $\mathrm{d}x$ is given by

$$\mathrm{d}W = F_{pull} \mathrm{d}x.$$

Here, $F_{pull}$ is the force needed to pull the dielectric out of the capacitor. Thus $F = -F_{pull}$. We can thus rewrite the force as

$$F = -\frac{\mathrm{d}W}{\mathrm{d}x}.$$

As derived in section 2.7 and applying definition 2.19, the energy stored in a capacitor is given by

$$W = \frac{1}{2} C V^2 = \frac{1}{2} \frac{Q^2}{C}.$$

Let's assume that $Q$ is held constant on the capacitor, thus our force becomes

$$F = \frac{1}{2} \frac{Q^2}{C^2} \frac{\mathrm{d}C}{\mathrm{d}x} = \frac{1}{2} V^2 \frac{\mathrm{d}C}{\mathrm{d}x}.$$

□

If we work with a plate capacitor, the capacitance is give by

$$C = \frac{\varepsilon w}{d} (\varepsilon_r l - \chi_e x).$$

Thus the force on the dielectric in a plate capacitor is given by

$$F = -\frac{\varepsilon \chi_e w}{2d} V^2.$$

# 5 Magnetostatics

So far, we have looked at a collection of source charges at rest. It is now time to look at charges in motion.

A classic experiment is to take two wires and run a current through them. It can be observed that the wires attract or repel each other depending on the direction of the current. This force clearly cannot be of electrostatic nature, as the wires themselves are neutral. This is the first appearance of a magnetic force and the **magnetic field** $\boldsymbol{B}$.

To measure the direction of the magnetic field one can simply use a compass.

## 5.1 The Lorentz Force Law

From experiments, one can deduce the following law.

**Theorem 5.1: Lorentz Force Law**

The force on a charge $Q$ moving with velocity $\boldsymbol{v}$ in a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F}_{\text{mag}} = Q(\boldsymbol{v} \times \boldsymbol{B}).$$

In the presence of both an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$, the total force on a charge $Q$ is thus given by

$$\boldsymbol{F} = Q(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}).$$

One implication of the Lorentz Force Law deserves special attention.

**Corollary 5.2:**

Magnetic forces do no work.

**Proof.** The work done can be calculated as

$$dW_{\text{mag}} = \boldsymbol{F}_{\text{mag}} \cdot \mathrm{d}\boldsymbol{l} = Q(\boldsymbol{v} \times \boldsymbol{B}) \cdot \boldsymbol{v} \mathrm{d}t.$$

As $\boldsymbol{v} \times \boldsymbol{B}$ is by definition perpendicular to $\boldsymbol{v}$, we see that the scalar product is zero. □

Our main goal now becomes to figure out how to calculate the magnetic field $\boldsymbol{B}$ and with that also how the electric field $\boldsymbol{E}$ is affected by moving charges.

### 5.1.1 Currents

In order to describe moving charges, we introduce the concept of a **Current**.

**Definition 5.3: Current**

The current $I$ in a wire is the charge per unit time that passes through a given point. It is measured in Amperes $\mathrm{A} = \mathrm{C/s}$.

Thus, a line charge traveling along a wire with velocity $v$ and charge density $\lambda$ will have a current $I = \lambda v$.[9] But actually, we should assign a direction to the current as velocity is a vector

$$\boldsymbol{I} = \lambda \boldsymbol{v}.$$

[9] Normally only one type of the charges are actually moving, but if both were moving, we would have $I = \lambda_+ \boldsymbol{v}_+ + \lambda_- \boldsymbol{v}_-$.

This notation might seem unsensible at the moment, but when we transfer to surface and volume currents, it will be more relevant.

Using this notation, we can calculate the magnetic force on a wire carrying a current.

**Theorem 5.4: Magnetic Force on a wire**

The force on a wire carrying a current $\boldsymbol{I}$ in a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{F}_{\text{mag}} = \int I(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}).$$

**Proof.** If we integrate the Lorentz Force Law over a wire, we get

$$\boldsymbol{F} = \int (\boldsymbol{v} \times \boldsymbol{B})\mathrm{d}q = \int (\boldsymbol{v} \times \boldsymbol{B})\lambda \mathrm{d}l = \int \boldsymbol{I} \times \boldsymbol{B}\mathrm{d}l.$$

As both the current $\boldsymbol{I}$ and the wire element $\mathrm{d}\boldsymbol{l}$ point in the same direction, we can rewrite this as

$$\boldsymbol{F}_{\text{mag}} = \int I(\mathrm{d}\boldsymbol{l} \times \boldsymbol{B}).$$

□

When considering something more complicated than a wire we need to introduce the concept of a current density.

**Definition 5.5: Surface current density**

Consider a "ribbon" of infinitesimal width $\mathrm{d}\boldsymbol{l}_\perp$, parallel to the flow of charges. If the current in the ribbon is $dI$, then the **Surface Current Density** $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}\boldsymbol{l}_\perp}.$$

In other words, $\boldsymbol{K}$ represents the current per unit width of the ribbon. If the surface charge density is $\sigma$ and the ribbon is moving with velocity $\boldsymbol{v}$, then the surface current density is given by

$$\boldsymbol{K} = \sigma \boldsymbol{v}.$$

In general, $K$ isn't constant so we need to integrate over the surface to get the magnetic force on it. This is given by

$$\boldsymbol{F}_{\text{mag}} = \iint \boldsymbol{K} \times \boldsymbol{B}\mathrm{d}a.$$[10]

Similarly, we define the **Volume Current Density** for charge flowing through a volume.

**Definition 5.6: Volume current density**

Consider a tube of infinitesimal cross section $\mathrm{d}a_\perp$, running parallel to the flow of charges. If the current in this tube is $\mathrm{d}\boldsymbol{I}$, then the volume current density $\boldsymbol{J}$ is defined as

$$\boldsymbol{J} = \frac{\mathrm{d}\boldsymbol{I}}{\mathrm{d}a_\perp}.$$

In other words, $\boldsymbol{J}$ represents the current per unit area. Again, the force on a volume current is given by

$$\boldsymbol{F}_{\text{mag}} = \iiint (\boldsymbol{v} \times \boldsymbol{B})\rho \,\mathrm{d}\tau = \iiint (\boldsymbol{J} \times \boldsymbol{B})\mathrm{d}\tau.$$

[10] Similarly to electric fields, B-fields suffer a discontinuity at surface currents. One thus has to uise the average field above and below the surface.

Lastly, the following theorem can be useful for some definitions.

**Theorem 5.7: Continuity equation**

The continuity equation for charge is given by

$$\nabla \cdot \boldsymbol{J} = -\frac{\partial \rho}{\partial t}.$$

It states that charge is locally conserved.

**Proof.** We start by integrating definition 5.6 over a surface. This gives us

$$I = \iint_{\mathcal{S}} \boldsymbol{J} \cdot \mathrm{d}a.$$

Applying the divergence theorem (1.21) we get

$$\oiint_{\mathcal{S}} \boldsymbol{J} \cdot \mathrm{d}a = \iiint_{\mathcal{V}} \nabla \cdot \boldsymbol{J}\mathrm{d}\tau.$$

As charge is conserved, whatever flows out of the surface must come at expense of what is inside. Thus

$$\iiint_{\mathcal{V}} \nabla \cdot \boldsymbol{J}\mathrm{d}\tau = -\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathcal{V}} \rho \mathrm{d}\tau = -\iiint_{\mathcal{V}} \frac{\partial \rho}{\partial t}\mathrm{d}\tau.$$

We conclude as this applies for all volumes $\mathcal{V}$. □

## 5.2 The Biot-Savart Law

Stationary charges produce electric fields, which are constant in time. Steady currents produce magnetic fields, which are constant in time. The theory of steady currents is called **magnetostatics**. Formally, we have steady currents if

$$\frac{\partial \rho}{\partial t} = 0 \quad \text{and} \quad \frac{\partial \boldsymbol{J}}{\partial t} = 0.$$

This also implies that the continuity equation becomes

$$\nabla \cdot \boldsymbol{J} = 0.$$

If we want to calculate the magnetic field produced by a steady current, we can use the following theorem

**Theorem 5.8: Biot-Savart Law**

The magnetic field produced by a steady current $I$ at a point $\boldsymbol{r}$ is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\int \frac{I\mathrm{d}\boldsymbol{l} \times \hat{\boldsymbol{\imath}}}{\imath^2}.$$

Where we integrate along the current flow and where $\mu_0$ is the permeability of free space.

This law, which is also empirical is the magnetic counterpart to Coulomb's Law. For surface and volume currents, the law is given by

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iint \frac{\boldsymbol{K}(\boldsymbol{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\mathrm{d}a'$$

and

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint \frac{\boldsymbol{J}(\boldsymbol{r}') \times \hat{\boldsymbol{\imath}}}{\imath^2}\mathrm{d}\tau'$$

Lastly note, that the superposition principle holds for magnetic fields just like it does for electric fields.

## 5.3 Divergence and Curl of Magnetostatic Fields

We now have developed the theory to derive the second Maxwell equation.

**Theorem 5.9: Divergence of a Magnetic Field**

The divergence of a magnetic field $\boldsymbol{B}$ is zero.

$$\nabla \cdot \boldsymbol{B} = 0.$$

**Proof.** Let us begin with the general form of the Biot-Savart Law

$$\boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{\boldsymbol{\imath}}}{\imath^2} \mathrm{d}\tau'.$$

This formula gives the magnetic field at a point $\boldsymbol{r} = (x, y, z)$ due to a current distribution $\boldsymbol{J}(x', y', z')$.

Notice that the integral is over the primed coordinates, whereas the divergence just as the curl of $\boldsymbol{B}$ is taken with respect to the unprimed coordinates. Applying the divergence to the Biot-Savart Law, we get

$$\nabla \cdot \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \nabla \cdot \left( \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{\boldsymbol{\imath}}}{\imath^2} \right) \mathrm{d}\tau'.$$

Let us look at the integrand in detail. We can apply the product rule

$$\nabla \cdot \left( \frac{\boldsymbol{J}(\boldsymbol{r'}) \times \hat{\boldsymbol{\imath}}}{\imath^2} \right) = \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \cdot (\nabla \times (\boldsymbol{J}(\boldsymbol{r'}))) - \boldsymbol{J}(\boldsymbol{r'}) \cdot \left( \nabla \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right).$$

However, the first term is zero as $J$ does not depend on unprimed variables. The second term is clearly zero as well, as the curl of $\hat{\boldsymbol{\imath}}/\imath^2$ is zero. Thus the integrand is zero and we conclude. □

**Theorem 5.10: Ampère's Law**

The curl of a magnetic field $\boldsymbol{B}$ is given by

$$\nabla \times \boldsymbol{B} = \mu_0 \boldsymbol{J}.$$

**Proof.** We again start by applying the curl to the Biot-Savart Law

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \nabla \times \left( \boldsymbol{J}(\boldsymbol{r'}) \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) \mathrm{d}\tau'.$$

The strategy once again is to look at the integrand and apply the corresponding product rule.[11]

$$\nabla \times \left( \boldsymbol{J}(\boldsymbol{r'}) \times \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = \boldsymbol{J}(\boldsymbol{r'}) \left( \nabla \cdot \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) - (\boldsymbol{J}(\boldsymbol{r'}) \cdot \nabla) \frac{\hat{\boldsymbol{\imath}}}{\imath^2}.$$

Let us look at the two terms seperately. The first term is simply

$$\nabla \cdot \left( \frac{\hat{\boldsymbol{\imath}}}{\imath^2} \right) = 4\pi \delta^3(\boldsymbol{\imath}).$$

The second term is a bit more complicated. The trick is to notice that we can switch $\nabla$ to $\nabla'$ at the cost of a minus sign as the derivative only acts on $\frac{\hat{\boldsymbol{\imath}}}{\imath^2}$. One can then show that for steady currents, this term is zero.

So in total, we have

$$\nabla \times \boldsymbol{B}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint \boldsymbol{J}(\boldsymbol{r'}) \left( 4\pi \delta^3(\boldsymbol{\imath}) \right) \mathrm{d}\tau' = \mu_0 \boldsymbol{J}(\boldsymbol{r}).$$

□

By applying stokes' theorem (1.22) to Ampère's Law, we can also derive the integral form of Ampère's Law

$$\oint_C \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{l} = \mu_0 I_{\mathrm{enc}},$$

where $I_{\mathrm{enc}}$ is the current flowing through the enclosed surface.

Similarly to Gauss' Law, Ampère's Law is always true, yet it is only useful if the symmetry of the problem allows us to pull $\boldsymbol{B}$ out of the integral.

The hardest thing about applying Ampère's Law is to find the direction of the magnetic field. However this has to be done once for all symmetries and can then be reused. The main symmetries are

- Infinite straight line: $B$ points radially outwards
- Infinite plane: $B$ points parallel to the plane but perpendicular to the current
- Infinite solenoids: $B$ points parallel to the axis of the solenoid inside and is zero far away.
- Toroidal coils: $B$ is circumferential inside the coil and zero outside.

## 5.4 Magnetic Vector Potential

Just as $\nabla \times \boldsymbol{E} = 0$ implied the existence of a scalar potential $\boldsymbol{V}$, $\nabla \cdot \boldsymbol{B} = 0$ implies the existence of a vector potential $\boldsymbol{A}$ in magnetostatics.

**Definition 5.11: Magnetic Vector Potential**

The **Magnetic Vector Potential** $\boldsymbol{A}$ is defined as

$$\boldsymbol{B} = \nabla \times \boldsymbol{A}.$$

With the added condition that $\nabla \cdot \boldsymbol{A} = 0$.

Plugging this definition into Ampère's Law (5.10) gives us

$$\nabla^2 \boldsymbol{A} = -\mu_0 \boldsymbol{J}.$$

Assuming that $\boldsymbol{J}$ goes to zero at infinity, we can read of the solution as

**Theorem 5.12: Magnetic Vector Potential**

The magnetic vector potential $\boldsymbol{A}$ at a point $\boldsymbol{r}$ due to a steady current distribution $\boldsymbol{J}$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \iiint_{\mathcal{V}} \frac{\boldsymbol{J}(\boldsymbol{r'})}{\imath} \mathrm{d}\tau'.$$

If we compare the $\boldsymbol{A}$ and $V$, we see that $\boldsymbol{A}$ is a vector potential, which is arguably less useful than a scalar potential. However, it is still a bit simpler than the biot-savart law.

Going back to the discontinuity at surface currents, we can see that the discontinuity is in tangential direction.

**Theorem 5.13: Discontinuity of the magnetostatic field**

For a surface current $K$, the magnetic field above and below correlate as follows

$$\boldsymbol{B}_{\mathrm{above}} - \boldsymbol{B}_{\mathrm{below}} = \mu_0 (\boldsymbol{K} \times \hat{n}).$$

## 5.5 Multipole expansion

Similarly to the electric potential, we can also expand the magnetic vector potential in a multipole expansion. For that we can use the taylor expansion we have found in

[11] For simplicity, the terms involving derivatives of $\boldsymbol{J}(\boldsymbol{r'})$ are omitted, as they don't depend to the unprimed coordinates.

theorem 3.8. With it we get the multipole expansion in magnetostatics.

**Theorem 5.14: Multipole Expansion of the Magnetic Vector Potential**

The magnetic vector potential $\boldsymbol{A}$ at a point $\boldsymbol{r}$ due to a current loop with current $I$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \oint (r')^n P_n(\cos\theta) \mathrm{d}\boldsymbol{l'}.$$

Once again, we call the first term the monopole term, the second term the dipole term and so on. By theorem 5.9, we know that the monopole term is zero. Let us thus concentrate on the dipole term. For this we define the magnetic dipole moment $\boldsymbol{m}$ of a current loop

**Definition 5.15: Magnetic Dipole Moment**

The magnetic dipole moment $\boldsymbol{m}$ of a current loop with current $I$ and area $\mathcal{A}$ is defined as

$$\boldsymbol{m} = I \iint_{\mathcal{A}} \mathrm{d}\boldsymbol{a'}.$$

Using this definition, we can rewrite the dipole term of the multipole expansion as follows

**Theorem 5.16: Magnetic Dipole Term of the Multipole Expansion**

The magnetic dipole term of the multipole expansion of the magnetic vector potential is given by

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{\boldsymbol{m} \times \hat{r}}{r^2}.$$

**Proof.** The dipole term of the multipole expansion is given by

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta \mathrm{d}\boldsymbol{l'} = \frac{\mu_0 I}{4\pi r^2} \oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}) \mathrm{d}\boldsymbol{l'}.$$

We can rewrite this using the identity

$$\oint (\hat{\boldsymbol{r}} \cdot \boldsymbol{r'}) \mathrm{d}\boldsymbol{l'} = -\hat{\boldsymbol{r}} \oint \mathrm{d}\boldsymbol{a'},$$

resulting in

$$\boldsymbol{A}_{\text{dip}}(\boldsymbol{r}) = -\frac{\mu_0 I}{4\pi r^2} \hat{\boldsymbol{r}} \oint \mathrm{d}\boldsymbol{a'}.$$

Plugging in the definition of the magnetic dipole moment concludes the proof. □

Similarly to an ideal electric dipole, a ideal magnetic dipole exists. It consists of a current loop with a constant current $I$. If the radius goes to zero and the current goes to infinity such that the magnetic dipole moment $\boldsymbol{m}$ stays constant, we get the ideal magnetic dipole. The field of an ideal magnetic dipole is given by

$$\boldsymbol{B}_{\text{dip}}(\boldsymbol{r}) = \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{m} \cdot \hat{\boldsymbol{r}})\hat{\boldsymbol{r}} - \boldsymbol{m}}{r^3}.$$

# 6 Magnetic Fields in Matter

## 6.1 Magnetization

When one asks the average person about magnetism, they will probably think of a magnet, which does not have any obvious connection to moving charges. However, if we zoom in on the atomic scale, we see that electrons are running around the nucleus of an atom as well as revolve around their own axix. This is a current, which produces a magnetic field.[12]. Ordinarily, due to the random orientation of the atoms, these magnetic fields cancel out. However, if we align these dipoles, we call the material **magnetized**.

Unlike electric polarization, magnetization does not always point in the direction of the applied field. In cases where it is parallel, we call the material **paramagnetic**. If it is antiparallel, we call it **diamagnetic**. Some special materials, called **ferromagnetic**, can even retain their magnetization after the external field is removed. These materials are the most complicated ones theoretically, and thus we will begin with paramagnetism and diamagnetism.

**Theorem 6.1: Torque on a magnetic dipole**

The torque $\boldsymbol{\tau}$ on a magnetic dipole $\boldsymbol{m}$ in a magnetic field $\boldsymbol{B}$ is given by

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}.$$

**Proof.** We will only prove the case of a rectangular loop with current $I$ as any non-rectangular loop can be decomposed into rectangular loops, where the inner parts cancel out.

We place the center of the loop at the origin and tilt it an angle $\theta$ from the $z$-axis to the $y$-axis. Meanwhile, the magnetic field $\boldsymbol{B}$ is parallel to the $z$-axis.

The forces on the sloping sides cancel out, meanwhile the forces on the horizontal sides are equal and opposite, but create a torque.

$$\boldsymbol{\tau} = 2\left(\frac{a}{2} F \sin\theta\right) \hat{\boldsymbol{x}}.$$

The force on one side is given by

$$\boldsymbol{F} = Ib\boldsymbol{B}.$$

Using definition 5.15, we can rewrite this as

$$\boldsymbol{\tau} = \boldsymbol{m} \times \boldsymbol{B}.$$

□

Once again, this is only true for uniform magnetic fields. If the field is not uniform, we have to also conside the force on the dipole, which is given by

$$\boldsymbol{F} = \nabla(\boldsymbol{m} \cdot \boldsymbol{B}).$$

The torque we found is responsible for aligning the dipole with the magnetic field which causes the effect of paramagnetism.

Electrons don't only have a spin, but also orbit around the nucleus. As the electron is flying really fast around the nucleus, we can think of it as a steady current. When placed in a magnetic field, the velocity of the electron will increase which causes the magnetic moment to decrease. This is the effect of diamagnetism. The details are not too relevant as this again is truly a quantum mechanical

[12] The real reasons comes from the spin of the electron, which is a quantum mechanical effect

effect. However, it is important to note that even tho diamagnetism is applicable to all materials, it is very weak and thus mostly negligible.

In a magnetic field, matter becomes **magnetized**. To describe this, the magnetization $\boldsymbol{M}$ is defined as

**Definition 6.2: Magnetization**

The magnetization $\boldsymbol{M}$ of a material is defined as the magnetic dipole moment per unit volume. It is given by

$$\boldsymbol{M} = \frac{\boldsymbol{m}}{V}.$$

Similarly as in section 4, we will first look at the field the magnetization produces, whilst not worrying about how it got there in the first place. Afterwards we combine this with the field producing the magnetization to get the total field.

## 6.2 The field of a magnetized object

The vector potential $\boldsymbol{A}$ of a magnetized object is given by

**Theorem 6.3: Vector Potential of a Magnetized Object**

The vector potential $\boldsymbol{A}$ of a magnetized object with magnetization $\boldsymbol{M}$ is given by

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \frac{\boldsymbol{J}_b(\boldsymbol{r'})}{\imath}\mathrm{d}\tau' + \frac{\mu_0}{4\pi}\oiint_{\mathcal{S}} \frac{\boldsymbol{K}_b(\boldsymbol{r'})}{\imath}\mathrm{d}a'.$$

Where $\boldsymbol{J}_b$ is the bound volume current density and $\boldsymbol{K}_b$ the bound surface current density as in definition 6.4.

**Proof.** As usual it is enligthening to start with a single dipole moment. By theorem 5.16, the vector potential is

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\frac{\boldsymbol{m}\times\hat{\boldsymbol{\imath}}}{\imath^2}.$$

Applying definition 6.2, we can integrate this over our object

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \frac{\boldsymbol{M}(\boldsymbol{r'})\times\hat{\boldsymbol{\imath}}}{\imath^2}\mathrm{d}\tau'.$$

Looking at the integrand, we can apply the identity

$$\nabla'\frac{1}{\imath} = \frac{\hat{\boldsymbol{\imath}}}{\imath^2}$$

to rewrite this as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\iiint_{\mathcal{V}} \boldsymbol{M}(\boldsymbol{r}')\times\left(\nabla'\frac{1}{\imath}\right)\mathrm{d}\tau'.$$

Integrating by parts and applying the corresponding product rule, we get

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\iiint_{\mathcal{V}}\frac{1}{\imath}(\nabla'\times\boldsymbol{M}(\boldsymbol{r'}))\mathrm{d}\tau' - \iiint_{\mathcal{V}}\nabla'\times\left(\frac{\boldsymbol{M}(\boldsymbol{r'})}{\imath}\right)\mathrm{d}\tau'\right).$$

The later can be expressed as a surface integral as

$$\boldsymbol{A}(\boldsymbol{r}) = \frac{\mu_0}{4\pi}\left(\iiint_{\mathcal{V}}\frac{1}{\imath}(\nabla'\times\boldsymbol{M}(\boldsymbol{r'}))\mathrm{d}\tau' - \oiint_{\mathcal{S}}\frac{1}{\imath}\boldsymbol{M}(\boldsymbol{r'})\times\mathrm{d}\boldsymbol{a'}\right).$$

The first term looks just like the vector potential of a steady volume current. Furthermore the second term looks like the potential of a steady surface current. We thus define the following with which we get our desired result. □

**Definition 6.4: Bound Currents**

The **Bound Volume Current Density** $\boldsymbol{J}_b$ and the **Bound Surface Current Density** $\boldsymbol{K}_b$ are defined as

$$\boldsymbol{J}_b = \nabla\times\boldsymbol{M}$$

and

$$\boldsymbol{K}_b = \boldsymbol{M}\times\hat{\boldsymbol{n}}.$$

To solve problems where the magnetization is given, it is useful to compute these bound currents and then use normal methods of magnetostatics to calculate the magnetic field.

Again, it is important to note that the bound currents are actual currents that arise and are not just a mathematical trick.

## 6.3 The auxiliary Field

We can now put together the field of the magnetization with the field of everything else which we call **free currents**. The free current might flow through the magnetized object or in case that object is a conductor even flow through it. The total current can be written as

$$\boldsymbol{J} = \boldsymbol{J}_b + \boldsymbol{J}_f.$$

With this we can define the **Auxiliary Field** $\boldsymbol{H}$ as

**Definition 6.5: Auxiliary Field**

The **Auxiliary Field** $\boldsymbol{H}$ is defined as

$$\boldsymbol{H} = \frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}.$$

For the $H$-field, we can derive the following theorem

**Theorem 6.6: Ampère's Law in Matter**

The curl of the auxiliary field $\boldsymbol{H}$ is given by

$$\nabla\times\boldsymbol{H} = \boldsymbol{J}_f.$$

**Proof.** We start from Ampère's Law (5.10) and plug in the bound and free currents

$$\frac{1}{\mu_0}\nabla\times\boldsymbol{B} = \boldsymbol{J}_b + \boldsymbol{J}_f = \boldsymbol{J}_f + (\nabla\times\boldsymbol{M}).$$

Combining the curls we get

$$\nabla\times\left(\frac{1}{\mu_0}\boldsymbol{B} - \boldsymbol{M}\right) = \boldsymbol{J}_f.$$

We now plug in the definition of the auxiliary field (6.5) which concludes the proof. □

Interestingly, $\boldsymbol{H}$ is a way more useful quantitiy than $\boldsymbol{D}$ in electrostatics. The reason for this is that if one builds an electromagnet, one reads the free current $I_f$ from the power supply which determines $\boldsymbol{H}$. On the other side, for the elctric displacement field $\boldsymbol{D}$, one does not place free charges on a capcitor, rather one places a voltage $V$ which in turn determines the electric field $\boldsymbol{E}$.

## 6.4 Linear and Nonlinear Materials

For most para- and diamagnetic materials, the magnetization is proportional to the applied magnetic field $\boldsymbol{B}$.

However, since typically $\boldsymbol{H}$ is the more useful quantity, the proportionality is written in terms of $\boldsymbol{H}$.

$$\boldsymbol{M} = \chi_m \boldsymbol{H},$$

where $\chi_m$ is the **magnetic susceptibility** of the material. This quantity is positive for paramagnetic materials and negative for diamagnetic materials.

Materials applying this linear relation are called **linear materials**. By the definition of the $\boldsymbol{H}$-field, we can rewrite this as

$$\boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}) = \mu_0 (1 + \chi_m) \boldsymbol{H} = \mu \boldsymbol{H}.$$

Here, $\mu = \mu_0(1 + \chi_m)$ is called the **magnetic permeability** of the material. The term in parenthesis is called the **relative permeability**.

### 6.4.1 Ferromagnetic Materials

Ferromagnetic materials are a special case of magnetized materials. They are not linear and dont require an external magnetic field to sustain their magnetization. Like paramagnetism, ferromagnetism is caused by the alignment of the spin of the electrons in the material. The huge difference comes from the fact that each dipole likes to align with its neighbors. The whole reason is quantum mechanical. It is important to know however that essentially, 100% of the dipoles in a ferromagnetic material are aligned with each other.

Now, why isn't any normal iron cube ferromagnetic? The reason is that this alignement only occurs in small regions called **domains**. Inside a domain, all dipoles are aligned, but each domain is randomly oriented.

If we now place a ferromagnetic material in a magnetic field, the torque can't exactly align all dipoles as they resist the change. However, at the boundaries between domains, the dipoles can more easily rotate by effectively changing the domain boundaries. If the external field is strong enough, one domain can take over the whole material, and the material is said to be **saturated**. If the external field is removed, the material will still be magnetized, as the domains are now aligned.

# 7 Electromagnetism

## 7.1 Electromotive Force

To make current flow, one has to push the charges. For most substances, the current density is proportional to the force per unit charge.

**Definition 7.1: Conductivity**

The **Conductivity** $\sigma$ of a material is defined as the ratio between the current density $\boldsymbol{J}$ and the force per unit charge $\boldsymbol{f}$.

$$\boldsymbol{J} = \sigma \boldsymbol{f}.$$

It is measured empirically.

Sometimes, one also sees the reciprocal of the conductivity, the **Resistivity** $\rho$. Applying the lorentz force law (5.1) gives the following

**Theorem 7.2: Ohm's Law**

The current density $\boldsymbol{J}$ in a conductor with conductivity $\sigma$ in an electric field $\boldsymbol{E}$ is given by

$$\boldsymbol{J} = \sigma \boldsymbol{E}.$$

**Proof.** This follows directly by plugging in the lorentz force law into the definition of the conductivity and using the fact that that for small velocities, $\boldsymbol{E} \gg \boldsymbol{v} \times \boldsymbol{B}$. □

The more familiar form of Ohm's Law $V = IR$ is derived from this by noticing that in order to double $V$, one has to double the charges on the electrodes which in turn doubles $E$. This then doubles $J$ (for ohmic materials) which doubles $I$.

For steady currents and uniform continuity, we see that

$$\nabla \cdot \boldsymbol{E} = \frac{1}{\sigma} \nabla \cdot \boldsymbol{J} = 0.$$

This implies that any unbalanced charge resides on the surface of the conductor.

When charges move around in a material, they are subject to collisions with atoms and other charges. This causes the current to be dissipated as heat. As charges are moving at a constant velocity, all the forces acting on them are converted to heat.

**Theorem 7.3: Joule Heating Law**

The power $P$ dissipated in a conductor with current density $\boldsymbol{J}$ is given by

$$P = VI = I^2 R.$$

**Proof.** As stated, the work done per unit charge is converted to heat. The work per unit charge is given by $V$ whilst the charge flowing per unit time is $I$. Thus the power is given by $VI$. □

Let us now look at a simple electric circuit, lets say a battery connected to a light bulb. A intriguing question is, how the current is the same all over the circuit, even though the only driving force is in the battery. The answer to this is found by noticing that if the current were not the same everywhere, charge would build up somewhere.

This would create an electric field which would even out the flow.

So in conclusion, we can split up the force per unit charge $\boldsymbol{f}$ into a source component $\boldsymbol{f}_\mathrm{s}$ and an electrostatic component $\boldsymbol{E}$.

$$\boldsymbol{f} = \boldsymbol{f}_\mathrm{s} + \boldsymbol{E}.$$

The source component may be caused by various things but is described by the **Electromotive Force** $\mathcal{E}$.

**Definition 7.4: Electromotive Force**

The **Electromotive Force (emf)** $\mathcal{E}$ is defined as

$$\mathcal{E} = \oint_C \boldsymbol{f}_\mathrm{s} \cdot \mathrm{d}\boldsymbol{l}.$$

As a warning, the electromotive force is not a force, rather it is the integral of a force per unit charge.

In an ideal source, the the net force per unit charge is zero so $\boldsymbol{f}_s = -\boldsymbol{E}$. The potential difference across an ideal source is thus

$$V = -\int_a^b \boldsymbol{E} \cdot \mathrm{d}\boldsymbol{l} = \int_a^b \boldsymbol{f}_s \cdot \mathrm{d}\boldsymbol{l} = \oint f_s \cdot \mathrm{d}l = \mathcal{E}.$$

We can expand the integral as $f_s = 0$ outside the source.

The most common source of an emf is a generator. These devices exploit motional emfs, which arise when one moves a wire through a magnetic field. A primitive example of this is a region of uniform magnetic field $\boldsymbol{B}$ perpendicular to the plane of the paper. If we move a loop of wire with velocity $\boldsymbol{v}$ to the right, the charges in the wire experience a lorentz force. (Figure 11)

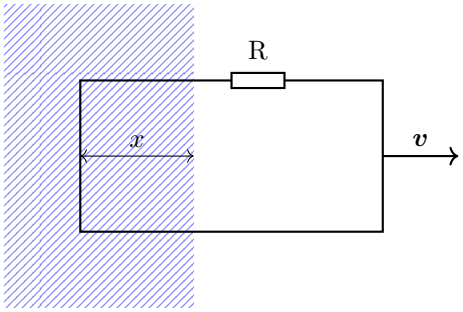


Figure 11: A wire loop moving through a magnetic field.

The charges in the vertical segment experience a magnetic force, whose vertical component $qvB$ drives current around the loop. As the horizontal segments cancel each other, we find

$$\mathcal{E} = \oint \boldsymbol{f}_{mag} \cdot \mathrm{d}\boldsymbol{l} = vBh,$$

where $h$ is the height of the loop.

A particularly beautiful way of expressing the emf by a moving loop is given includes the magnetic flux $\Phi_m$ through the loop.

**Definition 7.5: Magnetic Flux**

The **Magnetic Flux** $\Phi_m$ through a surface $\mathcal{S}$ is defined as

$$\Phi = \iint_\mathcal{S} \boldsymbol{B} \cdot \mathrm{d}\boldsymbol{a}.$$

If we calculate the time derivative of the magnetic flux through the loop, we get

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = -Bhv = -\mathcal{E}.$$

In fact, this is true for any loop moving in a magnetic field which we will accept without proof.

**Theorem 7.6: Flux rule**

The electromotive force $\mathcal{E}$ around a closed loop moving in a magnetic field is given by

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$

The flux rule is a handy shortcut but beware that it assumes a single wire loop, which can move, rotate, stretch or distort. But beware of switches, sliding contacts or extended conductors allowing current to flow in diffrent paths.

## 7.2 Electromagnetic Induction

Up to this point, we have seen current loops beeing pulled through magnetic fields. However, what if we move a magnet towards a stationary loop or let the magnetic field through a loop change in time? As it turns out, this also creates an emf in the loop. This phenomenon is called electromagnetic induction and was discovered by Michael Faraday in 1831.

**Theorem 7.7: Faraday's Law of Induction**

The electromotive force $\mathcal{E}$ around a closed loop in a time varying magnetic field is given by

$$\mathcal{E} = -\frac{\mathrm{d}\Phi}{\mathrm{d}t}.$$